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Metric Spaces and Topology HOMEWORK 8 Due: **Apr 18 (Tue)**

1. Prove the following theorem assuming X is a 2^{nd} countable topological space, however the theorem holds for all topological spaces.

Theorem (Banach category)**.** *Let X be a topological space. If $B \subseteq X$ is locally meagre*[1]*, then it is meagre.*

2. Let X be a topological space. For $B \subseteq X$, define the set $U(B)$ as the union of all open sets $U \subseteq X$ such that $U \Vdash B$. Prove:

(a) $U(B) \Vdash B$.

(b) B is Baire measurable if and only if $B \setminus U(B)$ is meagre if and only if $B =_* U(B)$.

(c) If X is Baire and B is Baire measurable, then $U(B)$ is the unique regular open[2] set that is $=_*$-equivalent to B.

REMARK: For a Baire measurable $B \subseteq X$, the set $U(B)$ is referred to as the **canonical representative** of the $=^*$-class of B.

3. [*Optional*] How about my grad school favourite: Clocks and Clouds by György Ligeti (conducted by Esa-Pekka Salonen)? I think this piece conveys perfectly Ligeti's point that music doesn't have to tell a story, it can just describe a state.

4. Let $T : X \to X$ be a transformation of a set X. Prove that a set $Y \subseteq X$ is E_T-invariant if and only if $Y = T^{-1}(Y)$.

5. For $k \in \mathbb{N}$ or $k = \mathbb{N}$, recall the (left) shift map $\sigma : k^{\mathbb{N}} \to k^{\mathbb{N}}$ given by $(x_n)_{n\in\mathbb{N}} \mapsto (x_{n+1})_{n\in\mathbb{N}}$.

(a) Prove that the **backward orbit** $\sigma^{-\mathbb{N}}(x) := \bigcup_{n\in\mathbb{N}} \sigma^{-n}(x)$ of each point $x \in X$ is dense.

(b) Conclude that for any nonempty open sets $U, V \subseteq k^{\mathbb{N}}$ there is $n \in \mathbb{N}$ such that $\sigma^{-n}(U) \cap V$ is nonempty open.

(c) Prove that the σ-preimage of an open dense set is open dense. Conclude that the σ-preimage of a meager set is meager.

(d) Conclude that σ is generically ergodic.

(e) Deduce that the graph G_σ does not admit a Baire measurable 2-colouring.

(f) Prove that in a generic infinite word $x \in k^{<\mathbb{N}}$ every finite word $w \in k^{<\mathbb{N}}$ appears infinitely many times. Conclude that for a generic $x \in X$, the **forward orbit** $\sigma^{\mathbb{N}}(x) := \{\sigma^n(x) : n \in \mathbb{N}\}$ is dense. (This part is not related to the previous parts.)

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[1] A set $B \subseteq X$ is **locally meagre** if each point of B admits an open neighbourhood U such that $B \cap U$ is meagre.

[2] A set U is called **regular open** if $\text{Int}(\overline{U}) = U$.

6. [*Optional*] Let Γ be a group that acts on a Polish space X by homeomorphisms[3]. The orbit of $x \in X$ is the set $[x]_\Gamma := \Gamma x := \{\gamma x : \gamma \in \Gamma\}$ and they form the classes of the **orbit equivalence relation** E_Γ. We say that the action $\Gamma \curvearrowright X$ is **generically ergodic** if so is E_Γ. Prove that the following are equivalent.

   (1) $\Gamma \curvearrowright X$ is generically ergodic.

   (2) Every E_Γ-invariant nonempty open set is dense.

   (3) For comeagerly many $x \in X$, the orbit $[x]_\Gamma$ is dense.

   (4) There is a dense orbit.

   (5) For every nonempty open sets $U, V \subseteq X$, there is $\gamma \in \Gamma$ such that $(\gamma U) \cap V \neq \emptyset$.

   Hint: For (2)⇒(3), take a countable basis $\{U_n\}_{n \in \mathbb{N}}$ and consider $\bigcap_n [U_n]_\Gamma$, where $[U]_\Gamma := \Gamma U := \bigcup_{\gamma \in \Gamma} \gamma U$.

7. For $k \in \mathbb{N}$ or $k = \mathbb{N}$, define the (left) shift map $\sigma : k^{\mathbb{Z}} \to k^{\mathbb{Z}}$ given by $(x_n)_{n \in \mathbb{Z}} \mapsto (x_{n+1})_{n \in \mathbb{Z}}$. Prove that a generic orbit is dense and hence σ is generically ergodic (by the previous question). Deduce that G_σ does not admit a Baire measurable 2-colouring.

   Remark: The map σ induces the shift action of $\mathbb{Z}$ on $k^{\mathbb{Z}}$ by $m \cdot (x_n)_{n \in \mathbb{Z}} := (x_{n+m})_{n \in \mathbb{Z}}$ for $m \in \mathbb{Z}$. More generally, every countable group Γ acts on k^Γ by shift: $\gamma \cdot (x_\delta)_{\delta \in \Gamma} := (x_{\delta\gamma})_{\delta \in \Gamma}$. One can again prove that a generic orbit of this action is dense and hence the action is generically ergodic.

8. Prove that in the Hamming graph $\mathcal{H}$ on $2^{\mathbb{N}}$, the set of neighbours $N_{\mathcal{H}}(M)$ of a meagre set $M \subseteq 2^{\mathbb{N}}$ is meagre.

9. [*Optional*] Prove:

   (a) $\mathcal{G}_0$ is acyclic.

   (b) $\mathcal{G}_0$ spans the Hamming graph, i.e. the $\mathcal{G}_0$-connectedness relation is the same as eventual equality $\mathbb{E}_0$.

   (c) $\mathcal{G}_0$ does not admit a Baire measurable countable colouring.

   Remark: Try to think what would happen in the argument if Baire measurability was replaced by measurability[4] instead. The 100% lemma becomes 99% and the argument breaks down. Indeed, it is a theorem of Ben Miller that $\mathcal{G}_0$ admits a measurable 3-colouring!

[3] You can think of Γ as a set of homoeomorphisms $X \to X$ that contains the identity automorphism and is closed under composition and inverses.

[4] The most natural (Haar) measure on $2^{\mathbb{N}}$ is the so-called fair coin flip measure μ, which assigns to each cylinder $[w]$ measure $2^{-|w|}$.