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Metric Spaces and Topology HOMEWORK 7 Due: **Mar 28 (Tue)**

1. Let X be a second-countable metric space (or just a topological space).

(a) Encode each open subset of X as a subset of $\mathbb{N}$, i.e. inject the collection of open sets into $\mathscr{P}(\mathbb{N})$. Deduce that there are at most continuum many open/closed sets.

(b) [*Optional*] Let $(\Lambda, <)$ be a well-order and let $(A_\lambda)_{\lambda \in \Lambda}$ be a strictly monotone (increasing or decreasing) sequence of open subsets of X. Prove that Λ is countable. Deduce the same statement but with "closed" instead of "open".

2. [*Optional*] **Construction of a Bernstein set.** Let X be a Polish space, say $X := \mathbb{R}$. We use AC to build a Bernstein set, i.e. a subset $B \subseteq X$ such that both B and B^c are of cardinality continuum but do not contain a homeomorphic copy of $2^{\mathbb{N}}$. Here are the steps:

(i) Due to the compactness of $2^{\mathbb{N}}$, the image of $2^{\mathbb{N}}$ under any continuous embedding is a perfect closed subset of X. Thus it is enough to build a set $B \subseteq X$ that does not contain any nonempty perfect closed subset.

(ii) By Problem 1(a), there are at most continuum many closed perfect subsets of X, so by AC, we can well-order them; more precisely, letting Λ be any continuum set, e.g. $2^{\mathbb{N}}$, AC gives (Zermelo's theorem) a well-ordering of $(\Lambda, <)$ such that for any $\beta \in \Lambda$, the set $\Lambda_{<\beta} := \{\alpha \in \Lambda\} : \alpha < \beta$ has cardinality *less than continuum*, and a sequence $(P_\lambda)_{\lambda \in \Lambda}$ enumerating all nonempty perfect closed subsets of X.

(iii) Using transfinite recursion, we build a sequence of pairs of points $(x_\lambda, y_\lambda)_{\lambda \in \Lambda}$ as follows: assuming $(x_\lambda, y_\alpha)_{\alpha < \beta}$ is built already for some $\beta \in \Lambda$, we define choose (using AC again) distinct points x_β, y_β from the set $P_\beta \setminus \{x_\alpha, y_\alpha : \alpha < \beta\}$. We can do so because by the Perfect Set Theorem, the set P_β is continuum, and hence so is $P_\beta \setminus \{x_\alpha, y_\alpha : \alpha < \beta\}$ because $\Lambda_{<\beta}$ is less than continuum.

(iv) The set $B := \{x_\lambda : \lambda \in \Lambda\}$ is continuum and does not contain a nonempty perfect closed set because it is missing at least one point y_λ from P_λ for each $\lambda \in \Lambda$.

3. In the proof of the Perfect Set Theorem, we defined a map $f : 2^{\mathbb{N}} \hookrightarrow X$ associated to the Cantor scheme. Prove that $f^{-1} : f(2^{\mathbb{N}}) \hookrightarrow 2^{\mathbb{N}}$ is continuous.

4. Let X be a perfect metric space (or just a topological space).

(a) Prove that open subsets of X are perfect.

(b) Prove that if $Y \subseteq X$ is perfect, then so is $\overline{Y}$. In particular, the closure of open sets (e.g. open balls) is perfect.

CAUTION: The closed balls of positive radius may not be perfect, as shown by the example in Lecture 12.

5. Prove that for any Polish space X, the partition $X = P \sqcup U$ into a perfect closed subset P and a countable open set U is unique. This P is called the **perfect kernel** of X.

6.* Show that $[0,1]$ does not admit a nontrivial[1] countable partition into closed intervals.

7. Cantor–Bendixson derivative and rank. Let X be a Polish space. The **Cantor–Bendixson derivative** of X, is the *closed* set X' resulting from removing the isolated points of X from X. Note that X' is itself a Polish space and it might have isolated points, so it makes sense to take its derivative again. By (transfinite) recursion, we define the iterated Cantor–Bendixson derivatives X^α, where α is an ordinal (think natural number), as follows:

$$\begin{aligned} X^0 &:= X, \\ X^{\alpha+1} &:= (X^\alpha)', \\ X^\lambda &:= \bigcap_{\alpha<\lambda} X^\alpha, \text{ if } \lambda \text{ is a limit.} \end{aligned}$$

Thus (X^α) is a decreasing transfinite sequence of closed subsets of X.

(a) [*Optional*] Prove that the iterated derivatives stabilize at a countable ordinal, i.e. there is a countable ordinal β such that $X^\beta = X^{\beta+1}$. The least such β is called the **Cantor–Bendixson rank** of X. Prove that X^β is the perfect kernel of X (this provides a more refined proof of the Cantor–Bendixson theorem).

(b) For each $n \in \mathbb{N}$, build a closed subset of $2^{\mathbb{N}}$ of Cantor–Bendixson rank n. Furthermore, build a (necessarily non-closed) subset of $2^{\mathbb{N}}$ of rank $\omega := \mathbb{N}$. Now build a closed subset of $2^{\mathbb{N}}$ of rank $\omega + 1$.

8. Prove that in any metric (or topological) space, finite unions of nowhere dense sets are nowhere dense. Thus, nowhere dense sets form an ideal.

9. Let X be a topological space, $Y \subseteq X$ be a subspace and $A \subseteq Y$. Prove:

(a) If A is nowhere dense (resp. meager) in Y, it is still nowhere dense (resp. meager) in X.

(b) If Y is open, then A is nowhere dense (resp. meager) in Y if and only if it is nowhere dense (resp. meager) in X.

10. Let G be a group equipped with a complete metric which makes the group multiplication operation $(g,h) \mapsto g \cdot h : G \times G \to G$ and inverse operation $g \mapsto g^{-1} : G \to G$ continuous. Prove:

(a) The closure of a subgroup is a subgroup.

(b) Every G_δ subgroup is closed.

[1] A partition $\mathcal{P}$ of a set X is *trivial* if $\mathcal{P} = \{X\}$.

Definition. Let X be a Baire space and let P be a property of points in X. We say that a **generic point in** X **satisfies** P (or P **holds generically**) if the set of $x \in X$ satisfying P is comeagre.

11. [*Optional*] Recall that $C([0,1])$ is a complete metric space with the uniform metric. Follow the steps below to prove that a generic function in $C([0,1])$ is nowhere differentiable.

(i) Prove that given $m \in \mathbb{N}$, each function $f \in C([0,1])$ can be approximated (in the uniform metric) by a piecewise linear function $g \in C([0,1])$, whose linear pieces (finitely many) have slope $\pm M$, for some $M \geqslant m$.

(ii) For each $n \geqslant 1$, let C_n be the set of all functions $f \in C([0,1])$, for which there is $x_0 \in [0,1]$ (depending on f) such that $|f(x) - f(x_0)| \leqslant n|x - x_0|$ for all $x \in [0,1]$. Show that C_n is nowhere dense using the fact that if g is as in step (i) with $m = 2n$, then some open neighborhood of g is disjoint from C_n.

12. [*Optional*] Prove that $\mathbb{N}^{\mathbb{N}}$ is homeomorphic to $\mathbb{R} \setminus \mathbb{Q}$.

Hint: Build a homeomorphism $\mathbb{R} \setminus \mathbb{Q} \to \mathbb{N}^{\mathbb{N}}$ using continued fraction expansion.