Last update: 2:04pm, Mar 13.

Metric Spaces and Topology HOMEWORK 6 Due: **Mar 17 (Fri)**

1. Prove directly via the diagonalization method that $2^{\mathbb{N}}$ and $[0,1)$ are uncountable.

2. Learn and present the proof of the Cantor–Schöder–Bernstein theorem, as written here in my notes.

3. Show that the injection $\mathbb{N}^{\mathbb{N}} \hookrightarrow 2^{\mathbb{N}}$ defined in class (via unary representation) is a continuous embedding (homeomorphism with its image), and explicitly describe its image.

Definition. A **basis** (or **base**) for a metric space (X,d) is a collection $\mathcal{U}$ of open sets such that every open set is a (potentially infinite) union of sets from $\mathcal{U}$. (For example, open balls form a basis.) A metric space is called **second countable** if it admits a countable basis.

4. Prove that a metric space is separable if and only if it is second countable.

Definition. A **cover** of a metric space (X,d) is a collection $\mathcal{U}$ of subsets of X whose union is X. A **subcover** is just a subcollection $\mathcal{V} \subseteq \mathcal{U}$ which is still a cover. An **open cover** is a cover consisting of open sets.

5. Let (X,d) be a second countable metric space. Prove:

(a) Every open cover of X has a countable subcover.

(b) Every basis has a countable sub-basis (a subcollection which is still a basis).

HINT: Use part (a).