Metric Spaces and Topology HOMEWORK 5 Due: **Mar 7 (Tue)**

Definition. For metric spaces (X, d_X), (Y, d_Y), a function $f : X \to Y$ is called **uniformly continuous** if for each $\varepsilon > 0$ there is a $\delta > 0$ such that f maps any set of diameter $\leqslant \delta$ to a set of diameter $\leqslant \varepsilon$, i.e. if $B \subseteq X$ has d_X-diameter $\leqslant \delta$ then $f(B)$ has d_Y-diameter $\leqslant \varepsilon$. Equivalently, $d_X(x_0, x_1) \leqslant \delta \implies d_Y(f(x_0), f(x_1)) \leqslant \varepsilon$ for all $x_0, x_1 \in X$.

1. Prove that the function $x \mapsto x$ is uniformly continuous on $\mathbb{R}$ but $x \mapsto x^2$ isn't.

2. Prove that uniformly continuous functions map Cauchy sequences to Cauchy sequences, i.e. if (x_n) is a Cauchy sequence in Y then $(f(x_n))$ is a Cauchy sequence in Y (in the notation of the above definition).

3. Let (X, d_X) and (Y, d_Y) be metric spaces, and let $D \subseteq X$. Suppose that (Y, d_Y) is complete. Prove that every uniformly continuous function $f : D \to Y$ admits a unique extension to a continuous function $\overline{f} : \overline{D} \to Y$, which is also uniformly continuous.

4. [*Optional*] Prove that the Cantor function (devil's staircase) defined in class (via ternary expansion) is uniformly continuous. Explain why its existence also follows abstractly from its definition on $\bigcup_{s \in 2^{<\mathbb{N}}} I_s = [0,1] \setminus \mathcal{C}$, where $\mathcal{C}$ is the Cantor set.

5. Let (X, d_X) and (Y, d_Y) be metric spaces. Prove that $C(X, Y)$ is a closed subset of Y^X equipped with the uniform extended metric d_u. Conclude that if (Y, d_Y) is complete, then the space $(BC(X, Y), d_u)$ of bounded continuous functions is also a complete metric space.

6. [*Optional*] The alternative proof of the existence of completion presented in class is actually my simplification of Kaplansky's proof. The following is Kaplansky's original proof, which doesn't deal with an extended metric, but uses a slightly less natural isometry $X \hookrightarrow BC(X, \mathbb{R})$ and the fact that $(BC(X, \mathbb{R}), d_u)$ is complete (the previous problem).

   Let (X, d) be a metric space. Fix a point $o \in X$ (think of it as the origin), and define a map $\iota : X \to \mathbb{R}^X$ by $x \mapsto \varphi_x$, where $\varphi_x(y) := d(x, y) - d(o, y)$. Prove that ι is an isometry into $BC(X, \mathbb{R})$ and conclude that $\overline{\iota(X)}$ is a completion of X.

7. [*Optional*] What musical composition do you consider most influential for the 20th century classical music? I think it's Stravinsky's "Rite of Spring." I simply love this piece, especially conducted by Esa-Pekka Salonen.

8. [*Optional*][1] Let (X, d) be a metric space and prove that for any dense G_δ set $D \subseteq X$, there is a function $f : X \to \mathbb{R}$ whose set of continuity points is exactly D.

[1] Thanks to Hayk Melkonyan for asking a version of this question.

9. Prove that the function $x \mapsto \frac{x}{1-x^2} : (-1,1) \to \mathbb{R}$ is a homeomorphism[2]. Conclude that $\mathbb{R} \equiv (a,b) \equiv [a,b]$ for any reals $a < b$.

Axiom of Choice (AC). *For each set $\mathcal{C}$ whose elements are nonempty sets, there is a* ***choice-function****, i.e. a function $f : \mathcal{C} \to \bigcup \mathcal{C}$ such that $f(S) \in S$ for each $S \in \mathcal{C}$. Here $\bigcup \mathcal{C} := \bigcup_{S \in \mathcal{C}} S$.*

10. Prove that each of the following statements is equivalent to AC:

(1) Every surjection has a **right-inverse**, i.e. if $f : A \to B$ is surjective, then there is a (necessarily injective) function $g : B \to A$ such that $f \circ g = \mathrm{id}_B$.

(2) Products of nonempty sets are nonempty, i.e. if I is a set and $(X_i)_{i \in I}$ is an I-indexed sequence of nonempty sets, then $\prod_{i \in I} X_i := \{(x_i)_{i \in I} : \forall i \in I \; x_i \in X_i\} \neq \emptyset$.

11. Let S be a set.

(a) Prove: if S is infinite then $\mathbb{N} \hookrightarrow S$. Pinpoint precisely where and how AC is used.

(b) Prove without AC: if S is Dedekind infinite then $\mathbb{N} \hookrightarrow S$.

12. Prove that if A is an infinite set and B is countable, then $A \equiv A \cup B$.

13. [*Optional*] Prove without AC: if a set A is countable, then $A^{<\mathbb{N}}$ is also countable.

14. [*Optional*] Prove that the set of algebraic reals is countable.

---

[2] In fact, this function is a diffeomorphism, unlike $x \mapsto \frac{x}{1-|x|}$.