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Metric Spaces and Topology HOMEWORK 4 Due: **Feb 28 (Tue)**

1. In the proof of the uniqueness of completion of a metric space (X,d), we defined a function f between two completions $\hat{X}$ and $\check{X}$. Prove:

(a) f is well-defined, i.e. does not depend on the choice of a Cauchy sequence approximating a given point $\hat{x} \in \hat{X}$.

(b) f is a bijective isometry.

2. Let (X,d) be a metric space. Prove that for any Cauchy sequences $(x_n),(x'_n) \subseteq X$, the sequence of distances $\left(d(x_n,x'_n)\right) \subseteq \mathbb{R}$ is Cauchy (with respect to the usual metric on $\mathbb{R}$).

Definition. We call sequences $(x_n),(x'_n)$ in a metric space (X,d) **equivalent**, and denote by $(x_n) \sim_d (x'_n)$, if $\lim_n d(x_n,x'_n) = 0$.

3. Let (X,d) be a metric space and let $(x^k)_{k\in\mathbb{N}^+}$ be a Cauchy sequence in $\hat{X} := \text{Cauchy}(X)/\sim_d$ with respect to the metric $\hat{d}$ defined in class. In particular, each x^k is a $\sim_d$-equivalence class of a Cauchy sequence $(x^k_n)_{n\in\mathbb{N}^+}$ in X.

(a) Prove that there is an increasing sequence $(n_k) \subseteq \mathbb{N}$ such that for each $k \in \mathbb{N}^+$,

(i) $\operatorname{diam}\left\{x^k_{n_k}, x^k_{n_k+1}, x^k_{n_k+2}, \ldots\right\} < \frac{1}{k}$;

(ii) $d(x^k_{n_\ell}, x^\ell_{n_\ell}) < \frac{1}{k}$ for all $\ell \geqslant k$.

(b) Conclude that each sequence $(x^k_n)_{n\in\mathbb{N}^+}$ has an equivalent sequence $(\tilde{x}^k_n)_{n\in\mathbb{N}^+}$ such that for each $k \in \mathbb{N}^+$,

(i) $\operatorname{diam}\left\{\tilde{x}^k_k, \tilde{x}^k_{k+1}, \tilde{x}^k_{k+2}, \ldots\right\} < \frac{1}{k}$;

(ii) $d(\tilde{x}^k_\ell, \tilde{x}^\ell_\ell) < \frac{1}{k}$ for all $\ell \geqslant k$.

4. For metric spaces $(X,d_X),(Y,d_Y)$, a function $f : X \to Y$, and a point $x_0 \in X$, prove that the following definitions of continuity at x_0 are equivalent:

(1) For every neighbourhood V of $f(x_0)$, the preimage $f^{-1}(V)$ is a neighbourhood of x_0.

(2) For every neighbourhood V of $f(x_0)$, there is a neighbourhood U of x_0 such that $f(U) \subseteq V$.

(3) For every $\varepsilon > 0$ there is $\delta > 0$ such that for all $x \in X$:

$$d_X(x,x_0) < \delta \implies d_Y(f(x),f(x_0)) < \varepsilon.$$

5. Prove:

(a) The function $f : 2^{\mathbb{N}} \to [0,1]$ defined as follows is a continuous surjection: $x := (x_n) \mapsto$ the real with binary representation $0.x_0x_1x_2\ldots$.

(b) The function $g : 2^{\mathbb{N}} \to \mathcal{C} \subseteq [0,1]$ defined as follows is a homeomorphism: $x := (x_n) \mapsto$ the real with ternary representation $0.\tilde{x}_0\tilde{x}_1\tilde{x}_2\ldots$, where $\tilde{x}_n := 2x_n$.

Hint: For continuity in parts (a) and (b), it is enough (why?) to consider preimages of intervals with endpoints having finite binary and ternary representation, respectively.

6. Prove carefully that Thomae's function is continuous at every irrational in $(0,1)$.

7. Let $(X,d_X),(Y,d_Y)$ be metric spaces and $f : X \to Y$ be an arbitrary function. Let $\mathrm{Cont}(f)$ denote the set of **continuity points** of f, namely those points in X at which f is continuous. Prove that $\mathrm{Cont}(f)$ is G_δ (i.e. countable intersection of open sets).

Hint: Consider the oscilation function $\mathrm{osc}_f : X \to \mathbb{R}$ defined by $x \mapsto \inf_{\delta>0} \mathrm{diam}\, f\big(B_\delta(x_0)\big)$.

8. [*Optional*] The graph of a function $f : X \to Y$ is the set

$$\mathrm{graph}(f) := \{(x,y) \in X \times Y : f(x) = y\}.$$

Prove that the graph of a continuous function $f : (X,d_X) \to (Y,d_Y)$ is a closed subset of $X \times Y$, where $X \times Y$ is equipped with the d_∞ metric, i.e. $d_\infty((x,y),(x',y')) := \max\{d_X(x,x'), d_Y(y,y')\}$. Try to give a purely topological proof without using metric or sequences.

9.* [*Optional*] A metric space (X,d) is called **completely metrizable** if it admits an equivalent metric $\tilde{d}$ which is complete. Suppose that (X,d) is a complete metric space. Prove that a subspace $Y \subseteq X$ is completely metrizable if and only if it is G_δ.

10. [*Optional*] Listen to Erik Satie's "Pièces Froides: Danses de travers." (all 3 parts) performed by Alexander Tharaud [spotify link], [youtube link by a different pianist, not as good]. I don't understand how this was written in 1897. If I heard that Brad Mehldau or Tigran Hamasyan wrote this yesterday, I wouldn't think twice about it.