Last update: 9:35am, Feb 18.

**Metric Spaces and Topology** | Homework 3 | Due: **Feb 21 (Tue)**

1. Let $(X,d)$ be a metric space and $Y \subseteq X$. The **boundary** of $Y$ is the set $\partial Y$ of all points $x \in X$ whose every neighbourhood intersects both $Y$ and $Y^c$.
   (a) Prove that $\partial Y = \overline{Y} \setminus \mathrm{Int}(Y)$ and conclude that $\partial Y$ is closed.
   (b) Prove that $Y$ is closed if and only if $Y \supseteq \partial Y$.
   (c) Determine the boundary of $\mathbb{Q}$ in the metric space $\mathbb{R}$ with the usual metric.

2. Let $(X,d)$ be a metric space and $Y \subseteq X$. Prove that $\mathrm{diam}(Y) = \mathrm{diam}(\overline{Y})$.

3. Let $(X,d)$ be a metric space and $(x_n) \subseteq X$. A point $x \in X$ is called a **subsequential limit** of $(x_n)$ if there is a subsequence of $(x_n)$ converging to $x$. Prove that the set of all subsequential limits of $(x_n)$ is closed.

   Hint: Let $(y_k)$ be a sequence of subsequential limits and assume that $y_k \to y$. Let $(x_{n_{k\ell}})_{\ell \in \mathbb{N}}$ be a subsequence converging to $y_k$. The indices $(n_{k\ell})$ form an infinite matrix ($k$ is the index of the row and $\ell$ is that of the column). Taking an appropriate "quasi-diagonal" of this matrix, we obtain a subsequence converging to $y$.

4. (a) Prove that Cauchy sequences are the same with respect to bi-Lipschitz equivalent metrics on a set $X$.
   (b) However, the Cauchy property is **not** preserved under equivalence of metrics. Indeed, construct a metric $d'$ equivalent to the usual metric $d$ on $[0,1)$ such that $d'$ has "fewer" Cauchy sequences than $d$, i.e. the $d'$-Cauchy sequences form a strict subset of the $d$-Cauchy sequences. Moreover, make sure that $d'$ is a complete metric on $[0,1)$, i.e. every $d'$-Cauchy sequence converges in $[0,1)$.
   (c) Nevertheless, show that any metric $d$ on a set $X$ has the same Cauchy sequences as its (equivalent) 1-bounded version $d' := \min\{d, 1\}$.

5. **Babylonian method of finding square roots.** The goal is to build a sequence of rationals that approximates $\sqrt{a}$, for some fixed positive $a \in \mathbb{Q}$. Take any $x_0 > 0$, and define the rest of the sequence recursively by

$$x_{n+1} := \frac{1}{2}\left(x_n + \frac{a}{x_n}\right). \qquad (*)$$

   Prove:
   (a) If $(x_n)$ converges in $\mathbb{R}$, then its limit is $\sqrt{a}$.
   (b) $x_n^2 \geqslant a$ for all $n \geqslant 1$.

   Hint: Rewrite equation $(*)$ as a quadratic equation in variable $x_n$ and consider its discriminant.

(c) The sequence $(x_n)$ is decreasing.

(d) Conclude that $(x_n)$ converges to $\sqrt{a}$ in $\mathbb{R}$.

**6.** Let $(X_n, d_n)_{n=1}^{\infty}$ be a sequence of metric spaces and assume that each $d_n \leqslant 1$. Let $X := \prod_{n\in\mathbb{N}} X_n$ be the Cartesian product of the sets $X_n$ and define $d_\infty : X \times X \to [0,1]$ by

$$d_\infty(x,y) := \sum_{n=1}^{0} 2^{-n} d_n\big(x(n), y(n)\big)$$

for $x, y \in X$.

(a) Prove that $d_\infty$ is a metric on $X$. We call this the **infinite product metric**.

(b) Prove that if each $d_n$ is complete, then so is $d_\infty$.

(c) For any set $A$, taking $X_n := A$ and $d_n$ the discrete metric on $A$, so $X = A^{\mathbb{N}}$. Prove that $d_\infty$ is bi-Lipschitz equivalent to the usual metric on $A^{\mathbb{N}}$. Conclude that $A^{\mathbb{N}}$ with the usual metric is a complete metric space. (We will give a much faster direct proof of this in class on Tuesday.)

**7.** Reward yourself with Steve Reich's Electric Counterpoint by Mat Bergström (or Pat Metheny, although he plays it a bit too fast for me). This is one of the truly revolutionary pieces of music.