Last update: 2:31pm, Feb 11.

Metric Spaces and Topology HOMEWORK 2 Due: **Feb 14 (Tue)**

1. Show that every open set U in $\mathbb{R}$ is a countable disjoint union of open intervals.

HINT: For countability, use the density of $\mathbb{Q}$. For disjointness, prove that each point $x \in U$ admits a $\subseteq$-maximum open interval $I_x \subseteq U$.

2. Let A be a countable nonempty set (alphabet). Consider the metric space $A^{\mathbb{N}}$, whose metric is defined the same way as for $2^{\mathbb{N}}$ and $\mathbb{N}^{\mathbb{N}}$. Prove that every open set U in $A^{\mathbb{N}}$ is a countable disjoint union of cylinders.

3. Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function. Prove that $\{(x, y) \in \mathbb{R}^2 : y < f(x)\}$ is an open set in $\mathbb{R}^2$.

Definition. Let X be a set. Metrics d_1 and d_2 on X are called **bi-Lipschitz equivalent**, denoted $d_1 \sim_L d_2$, if there are constants $\alpha, \beta > 0$ such that $\alpha d_1 \leqslant d_2 \leqslant \beta d_1$. Metrics d_1 and d_2 are called **equivalent**, denoted $d_1 \sim d_2$ if they induce the same open sets, i.e. the metric spaces (X, d_1) and (X, d_2) have the same open sets.

4. Let X be a set.

(a) For metrics d_1, d_2 on X, prove that $d_1 \sim_L d_2$ implies $d_1 \sim d_2$.

(b) For a metric d on X, let $d' := \min(d, 1)$ and prove that $d' \sim d$.

REMARK: However, if d is unbounded, like in the case of $\mathbb{R}$, then $d' \nsim_L d$, as was stated in Homework 1.

5.* [*Optional*] Prove that every infinite metric space has an infinite open set whose complement is also infinite.

6. [*Optional*] If you need a restart and motivation, Bartok's Violin Concerto No. 2 played by Patricia Kopatchinskaja never fails. (Warning: going through this amazing piece will require work, one can't be a lazy listener.)

7. A set $W \subseteq N^{<\mathbb{N}}$ is called **dense** if for each $w \in \mathbb{N}^{<\mathbb{N}}$ there is $w' \in N^{<\mathbb{N}}$ such that the concatenation $ww' \in W$.

(a) Prove that if $W \subseteq \mathbb{N}^{<\mathbb{N}}$ is dense then attaching 0s in the end of each word in W yields a dense set $W0^\infty := \{w000\cdots : w \in W\}$ in $\mathbb{N}^{\mathbb{N}}$.

(b) Does there exist a dense set in $\mathbb{N}^{<\mathbb{N}}$ containing exactly one word of each length? Prove your answer.

8. Let $\mathcal{C} \subseteq [0, 1]$ denote the Cantor set. Prove that $[0, 1] \setminus \mathcal{C}$ is dense open in $[0, 1]$.

9. Let (X, d) be a metric space and let $A, B \subseteq X$. Define the **distance** between A, B by

$$d(A, B) := \inf_{a \in A, b \in B} d(a, b).$$

We simply write $d(a, B)$ and $d(B, a)$ instead of $d(A, B)$ and $d(B, A)$ if $A = \{a\}$.

(a) For any $a \in X$, prove that $d(a, B) = 0$ if and only if $a \in \overline{B}$.

(b) Construct an example of a metric space (X,d) and disjoint closed subsets $A, B \subseteq X$ such that $d(A, B) = 0$.

(c) For each $r \geqslant 0$, call the set $B_r(A) := \{x \in X : d(A, x) < r\}$ the **open** r**-ball** around A. Prove that $B_r(A)$ is an open set.

(d) Prove that $\overline{A} = \bigcap_{n \in \mathbb{N}^+} B_{1/n}(A)$.

(e) Conclude that every closed set is G_δ (i.e. a countable intersection of open sets), and hence every open set is F_σ (i.e. a countable union of closed sets).