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**Metric Spaces and Topology** HOMEWORK 1 Due: **Feb 7 (Tue)**

**Definition.** For an interval $I \subseteq \mathbb{R}$, a function $f : I \to \mathbb{R}$ is called **convex** if $f(tx + (1-t)y) \leqslant tf(x) + (1-t)f(y)$ for all $x, y \in I$ and $t \in [0,1]$. In other words, the value of $f$ at a convex combination of two points is below the same convex combination of values of $f$ at those points. **Concave** is defined the same way using $\geqslant$ instead.

**1.** Let $f : [0,\infty) \to \mathbb{R}$ be a concave function such that $f(0) \geqslant 0$.

(a) Prove that $f$ is superlinear: $f(tx) \geqslant tf(x)$ for all $t \in [0,1]$ and $x \geqslant 0$.

(b) Prove that $f$ is subadditive: $f(x+y) \leqslant f(x) + f(y)$ for all $x, y \geqslant 0$.

HINT: Start with $f(x) + f(y)$, write $x$ and $y$ as constant multiples of $x + y$ and use part (a).

(c) Let $(X, d)$ be a metric space and suppose in addition (to being concave) that $f(0) = 0$, $f(x) > 0$ for all $x > 0$, and that $f$ is increasing (not necessarily strictly). Prove that the composition $f \circ d$ (i.e. the function $(x, y) \mapsto f(d(x, y))$) is also a metric on $X$.

**2.** Let $n \in \mathbb{N}^+$ (positive natural numbers) and $p > 0$. Define the $p$-norm $\|x\|_p$ and the $\infty$-norm $\|x\|_\infty$ of a vector $x := (x_1, \ldots, x_n) \in \mathbb{R}^n$ by

$$\|x\|_p := (|x_1|^p + \cdots + |x_n|^p)^{1/p}$$
$$\|x\|_\infty := \max_i |x_i|.$$

(a) Follow the hint below to prove Minkovski's inequality: $\|x + y\|_p \leqslant \|x\|_p + \|y\|_p$ for all $x, y \in \mathbb{R}^n$.

HINT: The proof for $p = \infty$ is straightforward. For $p \in [1, \infty)$, we will use the homogeneity (invariance under scalar multiplication) of $\|x + y\|_p \leqslant \|x\|_p + \|y\|_p$, an idea I learnt from Terry Tao. Dividing both sides by $\|x\|_p + \|y\|_p$ and raising to the $p^{\text{th}}$ power, it is enough to prove that

$$\|tx' + (1-t)y'\|_p^p \leqslant 1,$$

where $x' := \frac{x}{\|x\|_p}$, $y' := \frac{y}{\|y\|_p}$, and $t := \frac{\|x\|_p}{\|x\|_p + \|y\|_p}$. Prove this using the convexity of the function $\alpha \mapsto \alpha^p$ for each coordinate $i \in \{1, 2, \ldots n\}$.

(b) Deduce that $d_p(x, y) := \|x - y\|_p$ is a metric on $\mathbb{R}^n$.

**Definition.** Let $X$ be a set. Metrics $d_1$ and $d_2$ on $X$ are called **bi-Lipschitz equivalent**, $d_1 \sim_L d_2$, if there are constants $\alpha, \beta > 0$ such that $\alpha d_1 \leqslant d_2 \leqslant \beta d_1$. This means that for all $x, y \in X$,

$$\alpha d_1(x, y) \leqslant d_2(x, y) \leqslant \beta d_1(x, y).$$

**3.** Let $X$ be a set. Prove:

(a) Bi-Lipschitz equivalence of metrics on $X$ is indeed an equivalence relation, i.e. it is reflexive ($d \sim_L d$), symmetric ($d_1 \sim_L d_2 \Rightarrow d_2 \sim_L d_1$), and transitive ($d_1 \sim_L d_2 \wedge d_2 \sim_L d_3 \Rightarrow d_1 \sim_L d_3$).

(b) Metrics $d_1, d_2$ on $X$ are bi-Lipschitz equivalent if and only if there is a constant $\lambda > 0$ such that $\frac{1}{\lambda} d_1 \leqslant d_2 \leqslant \lambda d_1$.

(c) Show that if $d$ is a metric on $X$, then $d' \coloneqq \min(d, 1)$ is also a metric on $X$. Provide and prove a necessary and sufficient condition for $d \sim_L d'$.

**4.** Let $n \in \mathbb{N}^+$.

(a) Prove that $\|x\|_\infty \leqslant \|x\|_p \leqslant n^{1/p} \|x\|_\infty$ for all $x \in \mathbb{R}^n$ and all $p \in [1, \infty]$.

(b) Deduce that the metrics $d_p$ are all equivalent to each other for $p \in [1, \infty]$.

(c) Also deduce that $\lim_{p \to \infty} \|x\|_p = \|x\|_\infty$ for all $x \in \mathbb{R}^n$.

**5.** [*Optional*] Take a break and listen to Pyramid Song by Radiohead. What are your thoughts on the structure of this song?

**Definition.** A **pseudo-metric** on a set $X$ is the same as a metric, but the requirement that distinct points have a positive distance is omitted; namely, it is a function $d : X \times X \to [0, \infty)$ such that $d(x, x) = 0$, $d(x, y) = d(y, x)$, and $d(x, z) \leqslant d(x, y) + d(y, z)$ for all $x, y, z \in X$.

**6.** Let $d$ be a pseudo-metric on a set $X$ and define a binary relation on $X$ by $x \approx_d y :\Leftrightarrow d(x, y) = 0$ for all $x, y \in X$.

(a) Prove that $\approx_d$ is an equivalence relation on $X$.

(b) For $x \in X$, let $[x]_d$ denote the $\approx_d$-equivalence class of $x$, and let $X/\approx_d$ denote the quotient of $X$ by $\approx_d$, i.e. $X/\approx_d$ is the set of $\approx_d$-equivalence classes. Prove that $d'([x]_d, [y]_d) \coloneqq d(x, y)$ is a well-defined metric on $X/\approx_d$.

**7.** [*Optional*] Let $X$ be a set and $d : X \times X \to \mathbb{R}$ be a function satisfying $d(x, x) = 0$, $x \neq y \Rightarrow d(x, y) \neq 0$, and $d(z, x) \leqslant d(x, y) + d(y, z)$ for all $x, y, z \in X$. Prove that $d$ is a metric.

**8.** Recall that on the Cantor space $2^{\mathbb{N}}$, we defined the metric $d(x, y) \coloneqq 2^{-n(x,y)}$, where $n(x, y)$ is the least index $i \in \mathbb{N}$ such that $x(i) \neq y(i)$.

(a) Prove that $d$ is an **ultra-metric**, i.e. a metric with the following stronger version of the triangle inequality: $d(x, z) \leqslant \max\{d(x, y), d(y, z)\}$ for all $x, y, z \in 2^{\mathbb{N}}$.

(b) For any $x \in 2^{\mathbb{N}}$ and $n \in \mathbb{N}$, describe the open ball $B_{2^{-n}}(x)$ (draw a picture) and show that it is equal to the closed ball $\overline{B}_{2^{-(n+\epsilon)}}(x)$ for all $\epsilon \in (0, 1]$. (We stated this in lecture, but I want you to work this out in detail on your own.)

**9.** Note that for each $n \in \mathbb{N}^+$, the set $\{0, 1\}^n$ is a subset of $\mathbb{R}^n$. Letting $d_H$ denote the Hamming metric on $\{0, 1\}^n$, for which $p \in [0, \infty]$ is the identity map $x \mapsto x$ an isometry from $(\{0, 1\}^n, d_H)$ to $(\mathbb{R}^n, d_p)$? Prove your answer.