Metric Spaces and Topology HOMEWORK 12 Due: **May 19 (Fri)**

1. Let $X = \prod_{i \in I} X_i$ be a product of sets. Show that no set of the form $\bigcap_{i \in I_0} [i \mapsto V_i]$ with I_0 finite, is contained in a set of the form $\bigcap_{j \in I_1} [j \mapsto U_j]$ with I_1 infinite and $U_j \subsetneq X_j$ for each $j \in I_1$.

2. **Trees.** Let A be a set (alphabet) and recall that for $u, v \in A^{<\mathbb{N}}$, we write $u \subseteq v$ if u is an initial subword of v, i.e. $|u| < |v|$ and $v|_{|u|} = u$. A (set-theoretic) **tree** on A is a nonempty subset $T \subseteq A^{<\mathbb{N}}$ downward closed under $\subseteq$, i.e. whenever $v \in T$ and $u \subseteq v$ then $u \in T$. Denote by $[T]$ the set of its infinite branches, i.e.

$$[T] := \left\{x \in A^{\mathbb{N}} : x|_n \in T \text{ for each } n \in \mathbb{N}\right\}.$$

We equip $A^{\mathbb{N}}$ with the product topology, where A is given the discrete topology.

(a) Prove that $[T]$ is a closed subset of $A^{\mathbb{N}}$.

(b) For $u, v \in T$, call v a **successor** (or a **child**) of u if $v = ua$ for some $a \in A$. Call a vertex $u \in T$ a **leaf** if it has no successor in T. Call T **pruned** if it has no leaves.

For each set $Y \subseteq A^{\mathbb{N}}$, define a pruned tree T_Y on A so that $\overline{Y} = [T_Y]$. Thus, the closed subsets of $A^{\mathbb{N}}$ are exactly those of the form $[T]$ for some pruned tree T on A.

(c) We say that a tree T on A is **finitely branching** if each $u \in T$ has only finitely many successors.

Prove that if T is finitely branching then $[T]$ is compact.

(d) Conversely, for a pruned tree T, if $[T]$ is compact then T is finitely branching. Thus, the compact subsets of $A^{\mathbb{N}}$ are exactly those of the form $[T]$ for some finitely branching pruned tree T on A.

3. Let X be a topological space and E be an equivalence relation on X. Recall that the **quotient** X/E is the set of E-equivalence classes and let $\pi : X \twoheadrightarrow X/E$ be the **quotient map**, i.e. $x \mapsto [x]_E$, where $[x]_E$ denotes the E-equivalence class of x. The **quotient topology** on X/E is defined by declaring a set $U \subseteq X/E$ open if $\pi^{-1}(U)$ is open in X. In other words, this is the strongest topology for which π is continuous. Prove:

(a) This is indeed a topology.

(b) For any topological space Y, a map $f : X/E \to Y$ is continuous if and only if $f \circ \pi : X \to Y$ is continuous.

(c) The quotient topology is T_1 if and only if each E-equivalence class is closed.

(d) Let $E_{\mathbb{Z}}$ be the equivalence relation on $\mathbb{R}$ defined by being in the same $\mathbb{Z}$-coset, i.e. reals $x E_{\mathbb{Z}} y$ if $y - x \in \mathbb{Z}$. Considering the usual (Euclidean) topology on $\mathbb{R}$, the quotient $\mathbb{R}/\mathbb{Z} := \mathbb{R}/E_{\mathbb{Z}}$ is homeomorphic to the unit circle $S^1 \subseteq \mathbb{R}^2$.

4. Let $\mathcal{T}_0$ and $\mathcal{T}_1$ be topologies on a set X such that $\mathcal{T}_0 \subseteq \mathcal{T}_1$. Prove that if $\mathcal{T}_0$ is Hausdorff and $\mathcal{T}_1$ is compact, then $\mathcal{T}_0 = \mathcal{T}_1$. Deduce that in any chain of topologies $(\mathcal{T}_i)_{i \in I}$ on X, i.e.

$(I,<)$ is a linearly ordered set and $\mathcal{T}_i \subseteq \mathcal{T}_j$ for all $i<j$ in I, the indices $i \in I$ for which X_i is compact form an initial set[1] I_{cmp} of I, and the indices $j \in I$ for which X_j is Hausdorff form a terminal set[1] I_{Hsd} of I, and $|I_{\text{cmp}} \cap I_{\text{Hsd}}| \leqslant 1$, i.e. էրկու էրնեկ ամենաշատը մի տեղ ա լինում.

5. Prove that a subset of $\mathbb{R}^n$ (in any d_p metric, $1 \leqslant p \leqslant \infty$) is compact if and only if it is closed and bounded.

6. For each $n \in \mathbb{N}^+$, let $f_n : [0,1] \to 10 \coloneqq \{0,1,\dots,9\}$ be the function that maps each $x \in [0,1]$ to the n^{th} digit after '0.' in the decimal representation of x (pick an option to make the decimal representation unique). Prove that the sequence (f_n) does not have a pointwise convergent subsequence, so the product topology on $10^{[0,1]}$ is not sequentially compact.

7. [*Optional*] Prove that Tychonoff's theorem implies Axiom of Choice.

Hint: Given a sequence of sets $(X_i)_{i\in I}$, add a new point to each X_i obtaining a set $\tilde{X}_i$, and equip $\tilde{X}_i$ with the weakest topology (consisting of three open sets) in which this new point is isolated, turning $\tilde{X}_i$ into a compact space. Suppose a choice $(x_i)_{i\in I} \in \prod_{i\in I} X_i$ doesn't exist and build an open cover of $\prod_{i\in I} \tilde{X}_i$ with no finite subcover.

8. Let X be a first-countable topological space. Prove:

(a) Every point $x \in X$ admits a countable decreasing neighbourhood basis, i.e. there a neighbourhood basis (U_n) such that $U_n \supseteq U_{n+1}$ for all $n \in \mathbb{N}$.

(b) For any $Y \subseteq X$ and $x \in \overline{Y}$, there is a sequence $(y_n) \subseteq Y$ converging to x.

(c) [2]If X is compact then it is sequentially compact.

Hint: Consider the sets $K_n \coloneqq \overline{\{x_m : m \geqslant n\}}$.

(d) If in addition (to first countability) X is T_1 and a sequence $(x_n) \subseteq X$ does not have a convergent subsequence, then $\{x_n : n \in \mathbb{N}\}$ is closed.

(e) [3]If in addition (to first countability) X is normal, then X is sequentially compact if and only if every continuous real-valued function on X is bounded. Thus, for metrizable spaces compactness is equivalent to the boundedness of every continuous real-valued function.

Hint: For $\Leftarrow$, use part (d) and unbounded Tietze extension.

9. [*Optional*] Prove without using Axiom of Choice that a product of two compact spaces is compact.

Hint: Follow the outline written in Lecture 26 suggested by Hayk Karapetyan.

10. [*Optional*] Let X be a noncompact topological space. Prove using Zorn's lemma that there is an inclusion-maximal open cover of X that does not admit a finite subcover.

[1]An **initial** (resp. **terminal**) set J of $(I,<)$ is a downward (resp. upward) closed subset of I, i.e. if $j \in J$ and $i<j$ (resp. $j<i$) then $i \in J$.

[2]Thanks to Vahagn Hovhannisyan for asking this question!

[3]Thanks to Hayk Melqonyan for asking this question!