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Metric Spaces and Topology HOMEWORK 11 Due: **May 10 (Wed)**

1. For any set X, recall that $BC(X) := BC(X, \mathbb{R})$ denotes the metric space of bounded real-valued continuous functions with the uniform metric d_u. For each $f \in BC(X)$, let $\|f\|_u := d_u(f, 0)$, where 0 denotes the constant 0 function on X. For a sequence $(f_n) \subseteq BC(X)$, we say that the series $\sum_{n=0}^{\infty} f_n$
   (i) **uniformly converges** if the sequence $(\sum_{n=0}^{N} f_n)_N$ of partial sums uniformly converges, i.e. converges in the uniform metric d_u to some function $f \in BC(X)$.
   (ii) **absolutely converges** if the series (of real numbers) $\sum_{n=0}^{\infty} \|f_n\|_u$ converges.

   Prove that if the series $\sum_{n=0}^{\infty} f_n$ absolutely converges, then it uniformly converges.

2. Show that for any bounded interval $I \subseteq \mathbb{R}$ (including half-open), the Tietze extension theorem holds with I as the codomain, i.e. for a normal topological space X and a closed set $C \subseteq X$, a continuous function $f : C \to I$ admits a continuous extension $\overline{f} : X \to I$.

3. A topological space X is called **disconnected** if partitions into two nonempty open sets, i.e. $X = U \cup V$ where U, V are nonempty disjoint open (hence clopen) sets. Otherwise, we call it **connected**. Equivalently, X is connected if the only clopen sets are X and $\emptyset$. We say that a subset $Y \subseteq X$ is connected/disconnected if it is so in the relative topology.
   (a) Observe: A subspace $Y \subseteq X$ is disconnected if and only if there are (not necessarily disjoint) open (in X) sets $U, V \subseteq X$ such that $Y \subseteq U \cup V$, and $U \cap Y$ and $V \cap Y$ are nonempty and disjoint.
   (b) Prove: If $Y \subseteq X$ is connected, then so is $\overline{Y}$.
   (c) Let $(S_i)_{i \in I}$ be a collection connected subsets of X with nonempty pairwise intersections, i.e. $S_i \cap S_j \neq \emptyset$ for all $i, j \in I$. Prove that the union $\bigcup_{i \in I} S_i$ is connected.
   (d) Prove that for each point $x \in X$, there is $\subseteq$-maximum (i.e. largest) connected set $C_x \ni x$. Show that this set C_x is closed. C_x is called the **connected component** of x.
   (e) A space X is called **totally disconnected** if the connected component of each point $x \in X$ is $\{x\}$. Prove that zero-dimensional T_1 spaces are totally disconnected.
   (f) Prove that continuous functions map connected spaces to connected spaces, i.e. for topological spaces X, Y, if $f : X \to Y$ is continuous and X is connected, then $F(X)$ is connected (in the relative topology of Y).
   (g) Characterize the connected subsets of $\mathbb{R}$.
   (h) Deduce the **Intermediate Value Theorem**: Every continuous function $f : [a, b] \to \mathbb{R}$ admits all values between $f(a)$ and $f(b)$.

4. Prove the following theorem using the outline below:

   Theorem. *Compact Hausdorff spaces are normal.*

Proof-sketch. Let X be a compact Hausdorff space and let A, B be disjoint closed sets. Recall the proof that compact subsets of Hausdorff spaces are closed, and use its argument to show that X is T_3; in particular, for each point $x \in A$, there are disjoint open sets U_x and V_x such that $U_x \ni x$ and $V_x \supseteq B$. Switching the roles of A and B (treating the whole B as a "point"), use the same argument to show that there are disjoint open sets $U \supseteq A$ and $V \supseteq B$. □

5. In Lecture 23 Example (e), we outlined a proof that bounded closed intervals in $\mathbb{R}$ are compact. Complete this proof, filling in the details.

6. Show that in the cofinite topology on $\mathbb{R}$ (= Zariski topology), every subset is compact. Conclude that not all compact sets are closed, hence T_1 is not enough to make compact sets closed.

7. Prove that continuous functions map compact spaces to compact spaces, i.e. for topological spaces X, Y, if $f : X \to Y$ is continuous and X is compact, then $F(X)$ is compact (in the relative topology of Y). Deduce that every real-valued continuous function on a compact space is bounded and attains minimum and maximum values.

8. **Baire category for compact Hausdorff.** Prove that compact Hausdorff (hence normal) spaces are Choquet, and hence Baire.

9. [*Optional*] Prove the **almost perfect set property for compact Hausdorff spaces.** For any nonempty perfect compact Hausdorff space X, there is an injection $2^{\mathbb{N}} \hookrightarrow X$. (This injection need not be continuous though.) Pinpoint the use of Axiom of Choice, if any.