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Metric Spaces and Topology HOMEWORK 10 Due: **May 2 (Tue)**

1. Let $f : X \to Y$ be a function between topological spaces $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$. Let $\mathcal{T}'_X$ be the **refinement** of $\mathcal{T}_X$ obtained by **adjoining** the sets $f^{-1}(V)$, where $V \in \mathcal{T}_Y$, i.e., $\mathcal{T}'_X$ is the topology generated by $\mathcal{T}_X$ and sets of the form $f^{-1}(V)$. Prove:
   (a) f is not continuous if and only if $\mathcal{T}'_X$ is a strictly finer topology than $\mathcal{T}_X$.
   (b) On graph$(f) \subseteq X \times Y$, the restriction of the topologies $\mathcal{T}_X \times \mathcal{T}_Y$ and $\mathcal{T}'_X \times \mathcal{T}_Y$ coincide (even when $\mathcal{T}'_X$ is strictly finer than $\mathcal{T}_X$).
   (c) $(X, \mathcal{T}'_X)$ is homeomorphic to graph(f) in the product topology $\mathcal{T}_X \times \mathcal{T}_Y$. In particular, if f is continuous, then its domain is homeomorphic to its graph.

2. Let $X := \prod_{i \in I} X_i$ be the product of topological spaces X_i, $i \in I$. Prove that if a sequence $(x_n) \subseteq X$ converges to $x \in X$ in the product topology, then (x_n) converges to x **pointwise**, i.e. $\lim_n x_n(i) = x(i)$ for each $i \in I$.

   REMARK: The converse was proven in lecture.

3. Prove that the functions $f_n := x \mapsto x^n : [0,1] \to [0,1]$ pointwise converge to the function $f(x) := \begin{cases} 0 & \text{if } x \in [0,1) \\ 1 & \text{if } x = 1 \end{cases}$, but they don't converge to f uniformly (i.e. in the uniform metric).

4. Let A be a countable discrete topological space of at least two elements and let $X := A^{\mathbb{N}}$. Prove:
   (a) The box topology on $A^{\mathbb{N}}$ is discrete, in particular, not separable (although A is).
   (b) The product topology on $A^{\mathbb{N}}$ is induced by the usual metric $d(x,y) := 2^{-n}$ for distinct $x, y \in A^{\mathbb{N}}$, where $n \in \mathbb{N}$ is the least index at which x and y differ.

5. Prove that the countable product of separable topological spaces is separable (in the product topology).

6. [*Optional*] Consider the set $X := [0,1]^{\mathbb{N}}$, where $\mathbb{N}$ and $[0,1]$ are given the relative topology of $\mathbb{R}$.
   (a) Prove that the uniform topology on X is strictly coarser than the box topology.
   (b) Prove that the uniform topology on X is not separable.

7. Let X, Y, Z be topological spaces and let $f : X \to Y$ and $g : Y \to Z$. Recall that the composition $g \circ f : X \to Z$ is defined by $x \mapsto g(f(x))$. Prove:
   (a) If f is continuous at $x \in X$ and g is continuous at $f(x)$, then $g \circ f$ is continuous at x.

(b) If f is continuous and the restriction $g|_{f(X)}$ is continuous, then $g \circ f$ is continuous. In particular, compositions of continuous functions are continuous.

8. Let $X := \prod_{i \in \mathbb{N}} X_i$ be a countable product of second countable topological spaces and let $\mathcal{B}_i$ be a countable basis for X_i. Prove that the cylinders of the form

$$[i_1 \mapsto U_{i_1}, i_2 \mapsto U_{i_2}, \ldots, i_n \mapsto U_{i_n}],$$

where $n \in \mathbb{N}$, $(i_1, i_2, \ldots, i_n) \in \mathbb{N}^n$ and $U_{i_j} \in \mathcal{B}_{i_j}$, form a basis for X.

9. Let $X := \prod_{i \in \mathbb{N}} X_i$ be a countable product of metrizable topological spaces with a metric $d_i \leqslant 1$ for each $i \in \mathbb{N}$. Prove that the metric

$$d := \sum_{i \in \mathbb{N}} 2^{-i} d_i$$

on X induce the product topology on X.

Hint: To show that metric-open sets are product-open, it is enough (why?) to prove that for any ball $B_r(x)$ with $r > 0$, there is a product-open set $U \ni x$ and $U \subseteq B_r(x)$.

10. [*Optional*] Prove Urysohn's lemma for metric spaces more easily, using the metric.