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Homework 3 **Math 594/740: Topics in Ergodic...** **Due: Apr 28/29**

Lemma 1 (Rokhlin). *Let T be an aperiodic*[1] *pmp automorphism of a standard probability space (X, μ). For every $n \geqslant 1$ and $\varepsilon > 0$, there is a Borel $B \subseteq X$ such that the sets B, $T(B)$, $T^2(B)$, ..., $T^{n-1}(B)$ are pairwise disjoint and $\bigcup_{i=0}^{n-1} T^i(B)$ has measure $> 1 - \varepsilon$.*

Remark 2. The sequence B, $T(B)$, $T^2(B)$, ..., $T^{n-1}(B)$ is called a *Rokhlin tower* of height n.

1. Follow the steps below to prove the Rokhlin lemma (Lemma 1).

(i) Take a Borel complete E_T-section $M \subseteq X$ such that $\mu(M) < n\varepsilon$.

(ii) Take $B := \bigcup_{k \in \mathbb{N}} B_k$, where $B_k := \left\{x \in T^{kn}(M) : T^j(x) \notin M \text{ for all } 0 \leqslant j < k(n+1)\right\}$.

(iii) Show that the set $Z := X \setminus \bigcup_{i=0}^{n-1} T^i(B)$ is contained in $\bigcup_{i=1}^{n-1} T^{-i}(M)$, so its measure is $< \varepsilon$.

Hint: We are trying to tile each orbit by intervals of size n. Think of a single orbit, picturing it as a directed $\mathbb{Z}$-line, and colour the points in M red, the rest grey. Now colour the points in each B_k blue to understand what these sets are. Then colour all of $\bigcup_{i=0}^{n-1} T^i(B)$ blue to see which points are still left grey.

Theorem 3 (Luzin–Novikov). *Let X, Y be standard Borel spaces and $B \subseteq X \times Y$ be a Borel set. If the X-fibers of B are countable (i.e. B_x is countable for each $x \in X$), then $\mathrm{proj}_X(B)$ is Borel and $B = \bigcup_{n \in \mathbb{N}} \mathrm{graph}(f_n)$, where each $f_n : \mathrm{proj}_X(B) \to Y$ is a Borel function.*

Remark 4. This theorem says that countable-to-one Borel images of Borel sets are Borel. Moreover, it allows scanning the fibers in a Borel way: indeed, each $x \in X$ now has Borel names[2] for the elements in its fiber B_x, namely $f_0(x), f_1(x), \ldots$

2. Prove that any locally finite Borel graph G on a standard Borel space X admits a countable proper Borel vertex-colouring $c : X \to \mathbb{N}$.

Hint: Let (U_n) be an open basis for a Polish topology on X. Then each $x \in X$ admits a smallest n such that $x \in U_n$ but none of the G-neighbours of x are in U_n.

3. Let E be a CBER on a standard Borel space X.

For $n \geqslant 1$, let $[X]^n_E$ denote the set of all subsets $S \subseteq X$ of size n contained in a single E-class (i.e. the elements of F are all E-equivalent to each other). Denote $[X]^{<\mathbb{N}}_E := \bigcup_{n \geqslant 1} [X]^n_E$, the set of all E-equivalent finite nonempty subsets of X. Let $\mathcal{G}$ be the *intersection graph* on $[X]^{<\mathbb{N}}_E$, i.e. put an edge between the sets $S_0, S_1 \in [X]^{<\mathbb{N}}_E$ if $S_0 \cap S_1 \neq \emptyset$.

[1] Each T-orbit is infinite (hence looks like a $\mathbb{Z}$-line).

[2] Émile, Armand, ... (Forte Shinko's joke)

(a) Show that each $[X]_E^n$ can be viewed as a Borel subset of X^n and hence is standard Borel. Similarly for $[X]_E^{<\mathbb{N}}$.

(b) Note that $[X]_E^2$ is exactly the complete graphing $G_E := E \setminus \mathrm{Id}_X$ of E, and recall that the Feldman–Moore theorem is equivalent to the statement that the complete graphing G_E of E admits a countable proper Borel edge-colouring $G_E \to \mathbb{N}$. An edge-colouring of G_E is the same as a vertex-colouring of the intersection graph $\mathcal{G}$ restricted to $[X]_E^2$. Thus, the following generalizes the Feldman–Moore theorem:

Theorem 5 (Kechris–Miller)**.** *The intersection graph $\mathcal{G}$ on $[X]_E^{<\mathbb{N}}$ admits a countable proper Borel vertex-colouring $c : [X]_E^{<\mathbb{N}} \to \mathbb{N}$.*

Prove this theorem following these steps:

(i) Letting (f_n) be Borel functions (names) as in the Luzin–Novikov theorem applied to $B := E$, we record, for each $S \in [X]_E^{<\mathbb{N}}$ how each element of S refers to the others (so it's an $|S| \times |S|$ matrix of names). More precisely, fix a Borel linear order $<$ on X and write the elements of each $S \in [X]_E^{<\mathbb{N}}$ in the increasing order: $s_1 < s_2 < \dots s_k$. Map S to the $k \times k$ matrix $M_S \in \mathbb{N}^{k^2}$, whose (i, j) entry is the least $n \in \mathbb{N}$ such that $f_n(s_i) = s_j$.

(ii) The function $c_0 : S \mapsto M_S$ is a countable Borel colouring, but it may not be proper, i.e. there may be $S_0, S_1 \in [X]_E^{<\mathbb{N}}$ with $S_0 \cap S_1 \neq \emptyset$ having the same colour. However, prove that $\mathcal{G}$ restricted to each colour is locally finite, i.e. each $S \in [X]_E^{<\mathbb{N}}$ can only have finite-many $\mathcal{G}$-neighbours of the same c_0-colour.

(iii) Finish the proof using Question 2.

4. Let G be a locally countable Borel graph on a standard Borel space X. Call a set $Y \subseteq X$ *G-independent* if no two vertices in Y are connected by an edge in G. Prove that if G admits a countable proper Borel vertex-colouring then there is maximal independent set $Y \subseteq X$ that is Borel.

5. Let E be a CBER on a standard Borel space X and let $\Omega \subseteq [X]_E^{<\mathbb{N}}$ be a Borel set (think of Ω as a property). Prove that there is a maximal collection $\Omega' \subseteq \Omega$ that is Borel and the elements of Ω' are pairwise disjoint.

6. Let E be a pmp CBER on a standard probability space (X, μ). Prove in detail that if G is a Borel graphing of E that achieves the cost of E, then G is acyclic.

7. Prove in detail that a CBER E is hyperfinite if and only if it is induced by a Borel action of $\mathbb{Z}$.