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Homework 2 — **Math 594/740: Topics in Ergodic...** — **Due: Mar 21 (Mon)**

Definition 1. Let Γ be a countable semigroup (e.g. $\mathbb{N}$ or $\mathbb{Z}$) and let $\Gamma \curvearrowright (X, \mu)$ be a (right) pmp Borel action on a standard probability space. This induces a (left) action on of Γ on $L^2(X, \mu)$ by isometries $\gamma \cdot f(x) := f(x\gamma)$ (*Koopman representation*). We call $f \in L^2(X, \mu)$ *almost periodic* for this action if $\Gamma \cdot f := \{\gamma \cdot f : \gamma \in \Gamma\}$ is precompact[1] in $L^2(X, \mu)$.

1. Let T be a pmp transformation on (X, μ); in other words, we have a pmp action of $\mathbb{N}$. In class we proved that if T is weakly mixing then the only almost periodic functions in $L^2(X, \mu)$ are constants. Give a slightly cleaner proof of this using the equivalent definition of weak mixing where we take the usual limit (not that of averages) but we avoid a density 0 set.

Definition 2. Call a subset S of a countable group G *syndetic* if there is a finite subset $F \subseteq \Gamma$ such that $FS = \Gamma$; for $\Gamma := \mathbb{Z}$, this just means that S has bounded gaps.

2. Let $\Gamma \curvearrowright (X, \mu)$ be a pmp action of a countable group Γ. Prove that $f \in L^2(X, \mu)$ is almost periodic if and only if for each $\varepsilon > 0$, the set $\{\gamma \in \Gamma : \|\gamma \cdot f - f\|_2 < \varepsilon\}$ is syndetic.

Theorem 3 (Furstenberg Multiple Recurrence)**.** *Let $\mathbb{Z} \curvearrowright (X, \mu)$ be a pmp action, where 1 acts via the transformation T. For any $k \geqslant 1$ and any non-negative function $f \in L^\infty(X, \mu)$ with $\int_X f\, d\mu > 0$, there is $d \geqslant 1$ such that $\int_X f (T^d f)(T^{2d} f) \dots (T^{(k-1)d} f) d\mu > 0$. In particular, taking $f := \mathbb{1}_A$ for a measurable set $A \subseteq X$, we get $\mu(A \cap T^{-d} A \cap T^{-2n} A \cap \dots \cap T^{-(k-1)d} A) > 0$.*

3. Let $\mathbb{Z} \curvearrowright (X, \mu)$ be a pmp action, where 1 acts via the transformation T. Follow the steps below to prove the Furstenberg Multiple Recurrence theorem assuming f is almost periodic.

Assume without loss of generality that $\|f\|_\infty = 1$. Fix $\varepsilon > 0$ (to be specified later) and let $d \geqslant 1$ be such that $\|T^d f - f\|_2 < \varepsilon$ (there is a syndetic set of such d, remember?)

(a) Show that for every $j = 0, \dots, k-1$, $\|T^{jd} f - f\|_2 < k\varepsilon$. In particular, $\|T^{jd} f - f\|_1 < k\varepsilon$.

(b) Realize that for every $g \in L^\infty(X, \mu)$, pointwise multiplication by g is a Lipschitz operator, more precisely, for all $h \in L^1(X, \mu)$, $\|gh\| \leqslant \|g\|_\infty \|h\|_1$.

(c) Prove by induction on k that $\|f(T^d f)(T^{2d} f) \dots (T^{(k-1)d} f) - f^k\|_1 \leqslant O_k(\varepsilon)$, where $O_k(\varepsilon)$ means it is $C_k \varepsilon$, where C_k is a constant depending only on k.

Hint: Apply triangle inequality to $\|f(T^d f)(T^{2d} f) \dots (T^{(k-1)d} f) - f^k\|_1$ with the intermediate term $f(T^d f)(T^{2d} f) \dots (T^{(k-2)d} f) f$.

[1] Has compact closure. Equivalently, admits finite ε-nets.

(d) Conclude that $\int_X f(T^d f)(T^{2d} f)\dots(T^{(k-1)d} f\, d\mu = \int_X f^k + O_k(\varepsilon)$, and deduce the theorem.

Definition 4. A *countable Borel equivalence relation* (*CBER*) E on a standard Borel space X is an equivalence relation that is Borel (viewed as a subset of X^2) and each E-class is countable.

4. Let $\Gamma \curvearrowright X$ be a Borel action of a countable group Γ on a standard Borel space X. Verify that its orbit equivalence relation E_Γ is a CBER. You may use a theorem from Descriptive Set Theory saying that a function is Borel if and only if its graph is Borel.

Remark: There is basically nothing to do here, I just want you to absorb the definitions.

Theorem 5 (Feldman–Moore)**.** *Every CBER E is an orbit equivalence relation of a Borel action of a countable group. Moreover, this group can be taken to be generated by involutions*[2]*. In fact, there are Borel involutions $\gamma_n : X \to X$, $n \in \mathbb{N}$, such that $E = \bigcup_{n\in\mathbb{N}} \operatorname{graph}(\gamma_n)$.*

5. Let E be a CBER on a standard Borel space X.

(a) Show that the involutions γ_n in the Feldman–Moore theorem can be taken to be pairwise "disjoint", by which we mean that for all $n \neq m$, $\operatorname{graph}(\gamma_n) \cap \operatorname{graph}(\gamma_m) \subseteq \mathrm{Id}_X := \{(x,x) : x \in X\}$.

(b) Prove that the Feldman–Moore theorem (the "in fact" part) is equivalent to the statement that E admits a proper countable Borel edge-colouring, i.e. there is a Borel function $c : E \to \mathbb{N}$ such that the values of c on adjacent edges $(x,y),(y,z)$ are nonequal.

Hint: When going from an edge-colouring to involutions, use that a function is Borel if and only if its graph is a Borel set.

Definition 6. Let E be a CBER on a standard Borel space X. A set $M \subseteq X$ is called a *complete E-section* if it intersects every E-class. Call E *aperiodic* if each E-class is infinite.

Lemma 7 (Vanishing markers)**.** *Any aperiodic CBER E admits a vanishing*[3] *sequence $(M_n)_{n\in\mathbb{N}}$ of Borel complete E-sections.*

Remark 8. It would have been wonderful if we could choose a single point from each E-class, but such sets (given by Axiom of Choice) are typically not measurable. The next best thing (a poor person's Axiom of Choice) is a "small" complete E-section; indeed, the sets M_n for large enough n would have to be "small" since they eventually vanish. We refer to it as a *marker set* because it provides us with a "sparse" set of starting points/markers for algorithms that we can run locally on all E-classes at the same time, and the collective outcome would still be Borel.

[2]An *involution* is a group element whose inverse is itself, i.e. whose square is the identity.

[3]A sequence of sets is called *vanishing* if it is decreasing and the intersection is empty.

6. Follow the steps below to prove the vanishing markers lemma (Lemma 7).

(i) Suppose WLOG that $X = [0,1]$, and let $\ell : X \to X$ map each x to the least limit point of $[x]_E$. Note that ℓ is E-invariant (i.e. constant on each E-class), although many different E-classes might choose the same limit point.

(ii) Use the Feldman–Moore theorem to show that the map ℓ is Borel.

(iii) Let $M_n := \{x \in X : 0 < |x - \ell(x)| < 2^{-n}\}$. The sequence (M_n) works.