HOMEWORK 1 **Math 594/740: Topics in Ergodic...** **Due: Feb 18**

1. Let λ denote the Lebesgue measure on $[0,1)$ and identify $S^1 = \mathbb{R}/\mathbb{Z} = [0,1)$.

(a) Prove the following slight strengthening of the 99% lemma: For every positively-measured set $A \subseteq [0,1)$ and $\varepsilon > 0$, there are *arbitrarily small* open intervals $I \subseteq [0,1)$ such that $\frac{\lambda(A\cap I)}{\lambda(I)} > 1-\varepsilon$.

(b) Present a direct (without contradiction) proof that irrational rotation is ergodic by showing that the measure of an invariant positively-measured set is arbitrarily close to 1.

2. Let T be a pmp transformation on a standard probability space (X,μ). Prove that the following are equivalent:

(1) $\mu(A \cap T^{-n}B) \to \mu(A)\mu(B)$ as $n \to \infty$ for all sets A, B in some generating[1] algebra $\mathcal{A} \subseteq \mathcal{B}(X)$.

(2) T is mixing, $\mu(A \cap T^{-n}B) \to \mu(A)\mu(B)$ as $n \to \infty$ for *all* measurable sets $A, B \subseteq X$.

(3) For any $f, g \in L^2(X,\mu)$, $\langle f, T^n g\rangle \to \int f d\mu \int g d\mu$ as $n \to \infty$, where $\langle f, g\rangle$ denotes the usual inner product $\int f g d\mu$.

3. Let $T : 2^{\mathbb{N}} \to 2^{\mathbb{N}}$ be the *odometer* transformation, i.e. T maps a binary sequence x to the binary sequence obtained by thinking of x as a the binary representation of a number written in reverse and adding 1 to it, carrying it over if necessary. For example, $T(00110\ldots) := 10110\ldots$, $T(11101\ldots) := 00011\ldots$, and by convention, $T(11111\ldots) := 00000\ldots$. Let μ be the fair coin-flip measure on $2^{\mathbb{N}}$, i.e. $\mu := \nu^{\mathbb{N}}$, where $\nu := \frac{1}{2}\delta_0 + \frac{1}{2}\delta_1$.

Recall that the standard basic open sets of the product topology are of the form $[s] := \left\{x \in 2^{\mathbb{N}} : x|_{|s|} = s\right\}$ for $s \in 2^{<\mathbb{N}}$.

(a) Prove the 99% lemma for μ, namely: for any positively-measure $A \subseteq 2^{\mathbb{N}}$, there is a basic clopen set $[s]$ such that $\geqslant 99\%$ of $[s]$ is A, i.e. $\frac{\mu(A\cap[s])}{[s]} \geqslant 0.99$.

(b) Observe that for any $s, t \in 2^{<\mathbb{N}}$ of the same length, there is a $k \in \mathbb{Z}$ such that $T^k(sx) = tx$ for all $x \in 2^{\mathbb{N}}$, where by sx we meant the sequence s followed by x. In other words, powers of T can replace the beginning of an infinite sequence with any finite string.

(c) Deduce that T is ergodic (with respect to μ).

(d) Let $c : 2^{\mathbb{N}} \to 2^{\mathbb{N}}$ be the *conjugation* map, i.e. it maps a binary sequence x to a sequence obtained by switching the 0s of x to 1s and vice versa. Let $E := E_s \vee E_T$, i.e. E is the smallest equivalence relation containing E_s and E_T. Note that each E-class (except for that of $000\ldots$) contains exactly two E_T-classes. Prove that there

[1] By a *generating collection* of sets we mean one that generates the Borel σ-algebra of X.

is no measurable E_T-invariant set A that intersects every E-class in exactly one E_T-class; in other words, it is impossible to measurably choose one E_T-class out of every E-class.

4. Let T be a pmp transformation on a standard probability space (X, μ). Prove that the following are equivalent:

(1) T is ergodic.

(2) $\lim_{n\to\infty} A_n^T f = \int_X f \, d\mu$ a.e. for each $f \in L^1(X, \mu)$.

(3) $\lim_{n\to\infty} A_n^T \mathbb{1}_U = \mu(U)$ a.e. for each set U in some generating algebra $\mathcal{A} \subseteq \mathcal{B}(X)$.

(4) Same as (2) but the convergence is in L^1-norm, i.e. $\lim_{n\to\infty} \|A_n^T \mathbb{1}_U - \mu(U)\|_1 = 0$.

(5) $\lim_{n\to\infty} \|A_n^T \mathbb{1}_B - \mu(B)\|_1 = 0$ for each measurable set $B \subseteq X$.

Remark: I don't know a direct way of proving (3)$\Rightarrow$(2) without going through the full pointwise ergodic theorem. Do you?

5. Let (Y, ν) be a standard probability space and let $X := Y^{\mathbb{N}}$ and $\mu := \nu^{\mathbb{N}}$. Let $s : X \to X$ denote the shift transformation, i.e. $(x_n) \mapsto (x_{n+1})$.

(a) Prove that s is mixing, hence ergodic. (The proof is nearly identical to that for $Y := 2$ given in class.)

(b) Take $Y := \mathbb{R}$ and let ν be any Borel probability measure on $\mathbb{R}$. Applying the pointwise ergodic theorem to s with the function $f := \mathrm{proj}_0$ (i.e. the projection onto the 0th coordinate $(x_n) \mapsto x_0$), deduce the law of large numbers.

Hint: The only task here is to translate the probabilistic terminology (e.g. identically distributed random variables) to that of measure theory.

6. Prove the pointwise ergodic theorem for a general (not necessarily ergodic) pmp transormation T, i.e. with the conditional expectation in lieu of $\int f \, d\mu$.

7. [*Optional*] Let T be a pmp transformation on a standard probability space (X, μ). Prove that if $x \mapsto \nu_x : X \to P(X)$ is an T-ergodic decomposition of μ, then for each $f \in L^1(X, \mu)$ and a.e. $x \in X$, we have $\boldsymbol{E}(f | \mathcal{B}_T)(x) = \int_X f \, d\nu_x$.