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Math 595: Descriptive Set Theory HOMEWORK 5 **Due: Week of Dec 12**

1. A class $\mathbf{\Gamma}$ of sets is called *self-dual* if it is closed under complements, i.e. $\neg\mathbf{\Gamma} = \mathbf{\Gamma}$. Show that if $\mathbf{\Gamma}$ is a self-dual class of sets in topological spaces that is closed under continuous preimages, then for any topological space X there does not exist an X-universal set for $\mathbf{\Gamma}(X)$. Conclude that neither the class $\mathcal{B}(X)$ of Borel sets, nor the classes $\mathbf{\Delta}^0_\xi(X)$, can have X-universal sets.

2. Letting X be a separable metrizable space and $\lambda < \omega_1$ be a limit ordinal, put

$$\mathbf{\Omega}^0_\lambda(X) := \bigcup_{\xi<\lambda} \mathbf{\Sigma}^0_\xi(X) \ (= \bigcup_{\xi<\lambda} \mathbf{\Delta}^0_\xi(X) = \bigcup_{\xi<\lambda} \mathbf{\Pi}^0_\xi(X)).$$

(a) [*Optional*] Let Y be an uncountable Polish space and prove that there exists a set $P \in \mathbf{\Delta}^0_\lambda(Y \times X)$ that parameterizes $\mathbf{\Omega}^0_\lambda(X)$.

HINT: First construct such a set for $Y = \mathbb{N} \times 2^{\mathbb{N}}$. Then conclude it for $Y = 2^{\mathbb{N}}$ using the fact that the following functions are continuous: $()_0 : 2^{\mathbb{N}} \to \mathbb{N}$ and $()_1 : 2^{\mathbb{N}} \to 2^{\mathbb{N}}$ defined for $y \in 2^{\mathbb{N}}$ by

$$y = 1^{(y)_0} {}^\frown 0 ^\frown (y)_1.$$

Finally, conclude the statement for any Y using the perfect set property.

(b) Conclude that if X is uncountable Polish, then $\mathbf{\Delta}^0_\lambda(X) \supsetneq \mathbf{\Omega}^0_\lambda(X)$.

3. (a) Show that any Polish space admits a finer Polish topology that is zero-dimensional and has the same Borel sets, i.e. for a given Polish space $(X, \mathcal{T})$, there exists a zero-dimensional Polish topology $\mathcal{T}_0 \supseteq \mathcal{T}$ such that $\mathcal{B}(\mathcal{T}_0) = \mathcal{B}(\mathcal{T})$.

(b) Let Γ be a countable group and consider a Borel action of Γ on a Polish space $(X, \mathcal{T})$, i.e. each $g \in \Gamma$ acts as a Borel automorphism of X. Prove that there exists a Polish topology $\mathcal{T}_0 \supseteq \mathcal{T}$ with $\mathcal{B}(\mathcal{T}_0) = \mathcal{B}(\mathcal{T})$ that makes the action of Γ continuous. Moreover, $\mathcal{T}_0$ can be taken to be zero-dimensional.

4. Let X, Y be topological spaces and let $\mathrm{proj}_X : X \times Y \to X$ be the projection function. Prove the following statements:

(a) proj_X is continuous and open.

(b) proj_X does not in general map closed sets to closed sets, even for $X = Y = \mathbb{R}$.

(c) For $X = Y = \mathbb{R}$, proj_X maps closed sets to σ-compact (and hence F_σ) sets. More generally, images of F_σ sets under continuous functions from σ-compact to Hausdorff spaces are F_σ.

(d) **Tube lemma:** If Y is compact, then proj_X indeed maps closed sets to closed sets.

Hint: I would use the open cover definition of compact and show that for a closed $F \subseteq X \times Y$, every point $x \in X \setminus \mathrm{proj}_X(F)$ has an open neighborhood disjoint from $\mathrm{proj}_X(F)$. The "correct" solution should use nothing but definitions.

5. Show that the class of analytic sets is closed under countable unions.

6. Let X, Y be Polish and $f : X \to Y$ Borel. Show that for $A \subseteq f(X)$, if $f^{-1}(A)$ is Borel, then A is Borel relative to $f(X)$, i.e. there is a Borel $A' \subseteq Y$ such that $A = A' \cap f(X)$.

7. Let G be an (undirected) Borel graph on a standard Borel space X, i.e. $G \subseteq X^2$ is an irreflexive and symmetric Borel subset. Prove that if G is locally finite[1], then it admits a Borel (vertex) colouring $c : X \to \mathbb{N}$ with countably many colours. Pinpoint the use of the Luzin–Novikov theorem.

Hint: Separate a point from its neighbours by basic open sets.

8. Let X be Polish and let E be an analytic equivalence relation on X, i.e. E is an analytic subset of X^2.

(a) Show that for an analytic set A, its saturation $[A]_E := \{x \in X : \exists y \in A (x \, E \, y)\}$ is also analytic.

(b) Let $A, B \subseteq X$ be disjoint E-invariant analytic sets (i.e., $[A]_E = A$, $[B]_E = B$). Prove that there is an E-invariant Borel set D separating A and B, i.e., $D \supseteq A$ and $D \cap B = \emptyset$.

9. [*Optional*] Construct an example of a closed equivalence relation E on a Polish space X and a closed set $C \subseteq X$ such that the saturation $[C]_E$ is analytic but not Borel.

Remark: This shows that in part (a) of the previous question, "analytic" is the best we can hope for.

Hint: Take analytic $A \subseteq \mathbb{N}^{\mathbb{N}}$ that's not Borel and let $C \subseteq \mathbb{N}^{\mathbb{N}^2}$ be a closed set projecting down onto A. Define an appropriate equivalence relation E on $\mathbb{N}^{\mathbb{N}^2}$ (i.e. $E \subseteq \mathbb{N}^{\mathbb{N}^2} \times \mathbb{N}^{\mathbb{N}^2}$).

10. Let X be a Polish space. A function $s : \mathcal{F} \to X$ is called a selector if $s(F) \in F$ for every nonempty $F \in \mathcal{F}$. The goal of this question is to show that for every Polish space X, the Effros Borel space $\mathcal{F}$ admits a Borel selector.

(a) Show that $\mathcal{F}(\mathbb{N}^{\mathbb{N}})$ admits a Borel selector.

(b) By a previous homework question, there is a continuous open surjection $g : \mathbb{N}^{\mathbb{N}} \to X$. Prove that the map $f : \mathcal{F} \to \mathcal{F}(\mathbb{N}^{\mathbb{N}})$ defined by $F \mapsto g^{-1}(F)$ is Borel.

(c) Conclude that $\mathcal{F}$ admits a Borel selector.

11. [*Optional*] Let X be a Polish space. Show that $\mathcal{K}(X)$ is a Borel subset of $\mathcal{F}(X)$.

[1] Each vertex has only finitely many neighbours.

12. [*Optional*] Let X, Y be Polish spaces and let $f : X \to Y$ be a continuous function such that $f(X)$ is uncountable. Put

$$\mathcal{K}_f(X) = \{K \in \mathcal{K}(X) : f|_K \text{ is injective}\}.$$

(a) Note that, for a fixed countable basis $\mathcal{U}$ of X and for $K \in \mathcal{K}(X)$,

$$K \in \mathcal{K}_f(X) \iff \forall U_1, U_2 \in \mathcal{U} \text{ with } \overline{U_1} \cap \overline{U_2} = \emptyset [f(\overline{U_1} \cap K) \cap f(\overline{U_2} \cap K) = \emptyset].$$

Next, show that for fixed $U_1, U_2 \in \mathcal{U}$ with $\overline{U_1} \cap \overline{U_2} = \emptyset$ the set

$$\mathcal{V} = \left\{K \in \mathcal{K}(X) : f(\overline{U_1} \cap K) \cap f(\overline{U_2} \cap K) = \emptyset\right\}$$

is open in $\mathcal{K}(X)$, and hence $\mathcal{K}_f(X)$ is G_δ.

(b) Suppose that $f(U)$ is uncountable for each nonempty open $U \subseteq X$. Prove that $\mathcal{K}_f(X)$ is dense in $\mathcal{K}(X)$.

(c) Conclude that $f(X)$ contains a nonempty (compact) perfect set, so analytic sets have the Perfect Set Property.

13. (Fun problem, I think) Prove directly, **without using** Wadge's theorem or lemma or Borel determinacy, that any countable dense $Q \subseteq 2^{\mathbb{N}}$ is Σ^0_2-complete, by showing that Player 2 has a winning strategy in the Wadge game $G_W(A, Q)$ for any $A \in \Sigma^0_2(\mathbb{N}^{\mathbb{N}})$.