Last update: 9:29am, Nov 8.

Math 595: Descriptive Set Theory HOMEWORK 4 **Due: Nov 15-16**

1. Let G be a Polish group and let $H \leqslant G$ be a subgroup. Prove that H is Polish iff H is closed.

HINT: Consider H inside $\overline{H}$. What is the Baire category status (meager/nonmeager/comeager) of H in (the relative topology of) $\overline{H}$? If $H \subsetneq \overline{H}$, look at the cosets.

2. Let Γ be a group acting on a Polish space X by homeomorphisms (i.e. each element $\gamma \in \Gamma$ acts as a homeomorphism of X). A set $A \subseteq X$ is called invariant if $\gamma A = A$ for all $\gamma \in \Gamma$. The action $\Gamma \curvearrowright X$ is called *generically ergodic* if every invariant Baire measurable set $A \subseteq X$ is either meager or comeager. For a set $A \subseteq X$, denote by $[A]_\Gamma$ the saturation of A, namely $[A]_\Gamma = \bigcup_{\gamma \in \Gamma} \gamma A$.

Prove that the following are equivalent:

(1) $\Gamma \curvearrowright X$ is generically ergodic.
(2) Every invariant nonempty open set is dense.
(3) For comeagerly many $x \in X$, the orbit $[x]_\Gamma$ is dense.
(4) There is a dense orbit.
(5) For every nonempty open sets $U, V \subseteq X$, there is $\gamma \in \Gamma$ such that $(\gamma U) \cap V \neq \emptyset$.

HINT: For (2)$\Rightarrow$(3), take a countable basis $\{U_n\}_{n \in \mathbb{N}}$ and consider $\bigcap_n [U_n]_\Gamma$.

3. Show that if X, Y are second countable Baire spaces, so is $X \times Y$.

4. [*Optional*] Show that the Kuratowski–Ulam theorem fails if A is not Baire measurable by constructing a nonmeager set $A \subseteq \mathbb{R}^2$ (using AC) so that no three points of A are on a straight line. Construct this set A following these steps:

(i) Note that there are only continuum many F_σ subsets of $\mathbb{R}^2$, so take a transfinite enumeration $(F_\xi)_{\xi < 2^{\aleph_0}}$ of all *meager* F_σ sets.

(ii) Aim at recursively constructing a sequence $(a_\xi)_{\xi < 2^{\aleph_0}}$ of points in $\mathbb{R}^2$ by transfinite recursion so that for each $\xi < 2^{\aleph_0}$, $\{a_\lambda : \lambda \leqslant \xi\} \nsubseteq F_\xi$ and no three of the points in $\{a_\lambda : \lambda \leqslant \xi\}$ lie on a straight line.

(iii) To see that a choice of α_ξ is possible, note that $\{a_\lambda : \lambda < \xi\}$ generates $< 2^{\aleph_0}$ many lines and find (using Kuratowski–Ulam, ironically) a vertical line in $\mathbb{R}^2$ disjoint from $\{a_\lambda : \lambda < \xi\}$ and such that F_ξ is meager on that line.

5. Prove that in any topological group G, every nonmeager Baire measurable subgroup $H \leqslant G$ is actually clopen! In particular, if G is Polish and $H \leqslant G$ is Baire measurable, then H is clopen if and only if it has countable index in G.

6. Prove that if $f : (\mathbb{R}, +) \to (\mathbb{R}, +)$ is a Baire measurable group homomorphism, then f is just a multiplication by a constant, i.e. $f(x) = f(1)x$ for all $x \in \mathbb{R}$.

Hint: First show this for integers, then for rationals, etc.

7. For Polish spaces X, Y, a function $f : X \to Y$ is called *universally measurable* if the f-preimage of each Borel subset of Y is universally measurable. Prove that universally measurable functions are closed under composition, i.e. for Polish spaces X, Y, Z, if $f : X \to Y$ and $g : Y \to Z$ are universally measurable, then so is $g \circ f$.

8. Prove that the translation action of $\mathbb{Q}$ on $\mathbb{R}$ is generically ergodic and ergodic with respect to Lebesgue measure.

9. Let μ be any Bernoulli (coin-flip) measure on $2^{\mathbb{N}}$.

(a) Prove the **99% lemma** for μ, namely: any measurable set $B \subseteq 2^{\mathbb{N}}$ of positive measure admits a basic clopen set $[s]$, $s \in 2^{<\mathbb{N}}$, whose 99% is B. Show that $[s]$ can be taken to be arbitrarily μ-small (equivalently, s can be taken to be arbitrarily long).

(b) Deduce that the Hamming graph (equivalently, the corresponding group action $\bigoplus_{n\in\mathbb{N}} \mathbb{Z}/2\mathbb{Z} \curvearrowright 2^{\mathbb{N}}$) is μ-ergodic.

(c) Also observe that the Hamming graph (equivalently, the same group action as in (b)) is generically ergodic.

10. Let $C_0 := \left\{(x_n)_{n\in\mathbb{N}} \in [0,1]^{\mathbb{N}} : \lim_n x_n = 0\right\}$, and show that C_0 is in $\boldsymbol{\Pi}^0_3(X)$.