Math 595: Descriptive Set Theory HOMEWORK 3 **Due: Week of Oct 24**

1. Show that for any Polish space X there is a continuous open surjection $g : \mathbb{N}^{\mathbb{N}} \twoheadrightarrow X$ by constructing a sequence $(U_s)_{s \in \mathbb{N}^{<\infty}}$ of open subsets of X such that

(i) $U_\emptyset = X$

(ii) $\overline{U}_{s^\frown i} \subseteq U_s$

(iii) $U_s = \bigcup_i U_{s^\frown i}$

(iv) $\mathrm{diam}(U_s) < 2^{-|s|}$.

CAUTION: We don't require $U_{s^\frown i} \cap U_{s^\frown j} = \emptyset$ for $i \neq j$ (which makes your life easy), so the associated map g may not be injective.

2. The following steps outline a proof of the Baire category theorem for locally compact Hausdorff spaces.

(i) [*Optional*] Show that compact Hausdorff spaces are normal.

(ii) [*Optional*] Using part (i), prove that in locally compact[1] Hausdorff space X, for every nonempty open set U and every point $x \in U$, there is a nonempty precompact[2] open $V \ni x$ with $\overline{V} \subseteq U$.

(iii) Prove that locally compact Hausdorff spaces are Baire.

3. For topological spaces X, Y, a continuous map $f : X \to Y$ is called *category preserving* if f-preimages of meager sets are meager.

(a) Show that any continuous open map $f : X \to Y$ is category preserving (in fact, f-preimages of nowhere dense are nowhere dense). In particular, projections are category preserving.

(b) For topological spaces X, Y, if X is Baire, then, for a continuous map $f : X \to Y$, the following are equivalent:

(1) f is category preserving.

(2) f-preimages of nowhere dense sets are nowhere dense.

(3) f-images of open sets are somewhere dense.

(4) f-preimages of dense open sets are dense.

[1] A topological space is said to be **locally compact** if every point has a neighborhood basis that consists of precompact[2] open sets.

[2] **Precompact** sets are those contained in compact sets. For Hausdorff spaces, this is equivalent to having a compact closure.

4. (a) [*Optional*] Give an example of a function $f : \mathbb{R} \to \mathbb{R}$ that is continuous at every irrational but discontinuous at every rational.

   (b) Prove that there is no function $f : \mathbb{R} \to \mathbb{R}$ that is continuous at every rational but discontinuous at every irrational.

   Hint: Show that the set of continuity points of any function is G_δ.

   Remark: This is yet another example of a proof of nonexistence of an object based on the complexity of a set associated with it. I'm a fan.

5. [*Optional, but at least read it.*] Recall that $C([0,1])$ is a Polish space with the uniform metric. Show that a generic element of $C([0,1])$ is nowhere differentiable following the outline below.

   (i) Prove that given $m \in \mathbb{N}$, any function $f \in C([0,1])$ can be approximated (in the uniform metric) by a piecewise linear function $g \in C([0,1])$, whose linear pieces (finitely many) have slope $\pm M$, for some $M \geqslant m$.

   (ii) For each $n \geqslant 1$, let E_n be the set of all functions $f \in C([0,1])$, for which there is $x_0 \in [0,1]$ (depending on f) such that $|f(x) - f(x_0)| \leqslant n|x - x_0|$ for all $x \in [0,1]$. Show that E_n is nowhere dense using the fact that if g is as in (1) with $m = 2n$, then some open neighborhood of g is disjoint from E_n.

6. Let X be a perfect Polish space and show that a generic compact subset of X is perfect; in fact, the set $\mathcal{K}_p(X)$ of all perfect compact subsets of X is dense G_δ in $\mathcal{K}(X)$. Conclude that X has continuum many perfect compact subsets.

   Hint: Let $\mathcal{U}$ be a countable open basis for X, and for each $U \in \mathcal{U}$, let $\mathcal{K}_U$ denote the set of $K \in \mathcal{K}(X)$ such that either $K \cap U = \emptyset$ or $|K \cap U| \geqslant 2$. Prove that each $\mathcal{K}_U$ is dense G_δ.

7. Let $\mathcal{G}$ be the so-called *Hamming graph* on $2^{\mathbb{N}}$, namely, there is an edge between $x, y \in 2^{\mathbb{N}}$ exactly when x and y differ by one bit.

   (a) Prove that $\mathcal{G}$ is has no odd cycles and hence is bipartite (admits a 2-coloring). Pinpoint the use of AC.

   (b) Fix a coloring $c : 2^{\mathbb{N}} \to 2$ of $\mathcal{G}$ and let $A_i := c^{-1}(i)$ for $i \in \{0,1\}$. Consider the game where each player plays a finite nonempty binary sequence at each step and a run of the game is the concatenation of those finite sequences, thus an infinite binary sequence. Prove that this game with the payoff set A_0 is not determined by showing that if one of the players had a winning strategy, so would the other one.

   Hint: Steal the other player's strategy.

8. A *finite bounded game* on a set A is a game similar to infinite games, but the players play at most n number of steps before the winner is decided, for some fixed number $n \geqslant 1$ (say a million). More formally, the game is a tree $T \subseteq A^{<n}$, for some n, and the runs of the game are exactly the elements of the set $\mathrm{Leaves}(T)$ of all leaves of T, so the payoff set is a subset $D \subseteq \mathrm{Leaves}(T)$. Player I wins the run $s \in \mathrm{Leaves}(T)$ of the game iff $s \in D$. Consequently, Player II wins iff $s \in \mathrm{Leaves}(T) \setminus D$. All games that appear in real life are such games, e.g. chess (counting ties as a win for Player II).

Prove the determinacy of finite bounded games.

Hint: Let's write down what it means for Player I to have a winning strategy in this game, assuming for simplicity that n is even and that all of the runs of the game are of length exactly n:

$$\exists a_1 \forall a_2 ... \exists a_{n-1} \forall a_n ((a_1, ..., a_n) \in D).$$

What happens when you negate this statement?

9. A *finite game* on a set A is a game similar to infinite games, but the players play only finitely many steps before the winner is decided. More formally, it is a (possibly infinite) tree $T \subseteq A^{<\mathbb{N}}$ that has no infinite branches, and the set of runs is $\mathrm{Leaves}(T)$, so the payoff set is a subset $D \subseteq \mathrm{Leaves}(T)$. Player I wins the run $s \in \mathrm{Leaves}(T)$ of the game iff $s \in D$. Consequently, Player II wins iff $s \in \mathrm{Leaves}(T) \setminus D$.

(a) Prove the determinacy of finite games.

Hint: Call a position $s \in T$ determined, if from that point on, one of the players has a winning strategy. Thus, no player has a winning strategy in the beginning iff $\emptyset$ is undetermined. What can you say about extensions of undetermined positions?

(b) Conclude the determinacy of clopen infinite games. (These are games with runs in $A^{\mathbb{N}}$ and the payoff set a clopen subset of $A^{\mathbb{N}}$.)

10. [*Optional*] In ZF (in particular, don't use AC or $\neg$AD), define a game with rules $G(T, D)$ on the set $A = \mathscr{P}(\mathbb{N}^{\mathbb{N}})$ (i.e. define a pruned tree $T \subseteq A^{<\mathbb{N}}$ and a set $D \subseteq A^{\mathbb{N}}$), so that ZF+$\neg$AD implies that this game is undetermined. In other words, you have to define the tree T and the payoff set D without using $\neg$AD, but then prove that the game $G(T, D)$ is undetermined using $\neg$AD.

Hint: Note that besides playing subsets of $\mathbb{N}^{\mathbb{N}}$, players can also play natural numbers in the sense that $\mathbb{N} \hookrightarrow \mathscr{P}(\mathbb{N}^{\mathbb{N}})$ by $n \mapsto \{(n)_{i \in \mathbb{N}}\}$.

11. Prove that a topological group G is Baire iff G is nonmeager.

12. Let X be a topological space and $A \subseteq X$. Prove:

(a) $U(A)$ is regular open, i.e. it is equal to the interior of its closure.

(b) If moreover X is a Baire space and A is Baire measurable, then $U(A)$ is the unique regular open set U with $A =^* U$.