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Math 595: Descriptive Set Theory HOMEWORK 2 **Due: the week of Sep Oct 3**

1. [*Optional*] Using the outline below, prove the following:

Proposition. *Let (X, d) be a metric space. The following are equivalent:*

(1) *X is compact.*

(2) *Every sequence in X has a convergent subsequence.*

(3) *X is complete and totally bounded.*

(4) *X is separable and every decreasing sequences of nonempty closed sets has an intersection.*

In particular, compact metrizable spaces are Polish.

(1) $\Rightarrow$ (2): For a sequence $(x_n)_n$, let K_m be the closure of the tail $\{x_n\}_{n \geqslant m}$ of the sequence and use the intersection-of-closed sets version of the definition of compactness.

(2) $\Rightarrow$ (3): For total boundedness, fix an $\varepsilon > 0$ and start constructing an ε-net F by adding elements to your F that are not yet covered by $B(F, \varepsilon)$. For completeness, note that if a subsequence of a Cauchy sequence converges, then so does the entire sequence.

(3) $\Rightarrow$ (4): Separability follows from total boundedness, see Question **??**. Let (K_n) be a decreasing sequences of nonempty closed sets and $\varepsilon_n \to 0$. There is a finite collection $\mathcal{B}_0$ of balls or radius ε_0 that covers K_0. One of these balls has to intersect infinitely-many K_n.

(4) $\Rightarrow$ (1): Separability implies Lindelöf, i.e. every open cover has a countable subcover. Every countable open cover having a finite subcover is equivalent to every countable collection of closed sets with the finite intersection property having a nonempty intersection.

2. Let X be a compact metric space and Y be a separable complete metric space. Let $C(X, Y)$ be the space of continuous functions from X to Y equipped with the uniform metric, i.e. for $f, g \in C(X, Y)$,

$$d_u(f, g) = \sup_{x \in X} d_Y(f(x), g(x)).$$

(a) Prove that $C([0,1], \mathbb{R})$ is a separable complete metric space, hence Polish.

(b) [*Optional*] Prove more generally that $C(X, Y)$ is a separable complete metric space, hence Polish, for all X, Y as above.

HINT 1: Proving separability is tricky. Note that by uniform continuity,

$$C(X, Y) = \bigcup_n A_{n,m},$$

for every $n \in \mathbb{N}$, where

$$A_{n,m} = \{f \in C(X,Y) : \forall x,y \in X(d_X(x,y) < 1/n \Rightarrow d_Y(f(x),f(y)) < 1/m)\}.$$

Realize that it is enough to show that for any $n,m \in \mathbb{N}$, there is a countable $B_{n,m} \subseteq A_{n,m}$ such that for any $f \in A_{n,m}$ there is $g \in B_{n,m}$ with $d_u(f,g) < 3/m$. Now fix n,m and try to construct $B_{n,m}$; when doing so, don't try to *define* each function in $B_{n,m}$ by hand as you would maybe do in the case $X = [0,1]$; instead, carefully *pick* them out of functions in $A_{n,m}$.

Hint 2: This is Theorem 4.19 in Kechris's "Classical Descriptive Set Theory".

3. Show that Hausdorff metric on $\mathcal{K}(X)$ is compatible with the Vietoris topology.

4. Let (X,d) be a metric with $d \leqslant 1$. For $(K_n)_n \subseteq \mathcal{K}(X) \setminus \{\emptyset\}$ and nonempty $K \in \mathcal{K}(X)$:

(a) $\delta(K,K_n) \to 0 \Rightarrow K \subseteq \underline{\mathrm{T}\lim}_n K_n$;

(b) $\delta(K_n,K) \to 0 \Rightarrow K \supseteq \overline{\mathrm{T}\lim}_n K_n$.

In particular, $d_H(K_n,K) \to 0 \Rightarrow K = \mathrm{T}\lim_n K_n$. (We showed in class that the converse fails.)

5. Let (X,d) be a metric space with $d \leqslant 1$. Then $x \mapsto \{x\}$ is an isometric embedding of X into $\mathcal{K}(X)$.

6. Prove that if a topological space X is nonempty perfect, then so is $\mathcal{K}(X) \setminus \{\emptyset\}$.

7. Let X be a nonempty perfect Polish space and let Q be a countable dense subset of X. Show that Q is F_σ but not G_δ. In particular, $\mathbb{Q}$ is not Polish (in the relative topology of $\mathbb{R}$).

8. (Fun problem) Show that $[0,1]$ does not admit a countable nontrivial[1] partition into closed intervals. (I don't want to spoil it for you, but please ask me for a hint if you're stuck.)

9. A *topological group* is a group with a topology on it so that group multiplication $(x,y) \to xy$ and inverse $x \to x^{-1}$ are continuous functions. Show that a countable topological group is Polish if and only if it is discrete.

10. [*Optional*]

(a) Let X be a Polish space. Show that if $K \subseteq X$ is countable and compact, then its Cantor–Bendixson rank $|K|_C$ is not a limit ordinal.

(b) For each nonlimit ordinal $\alpha < \omega_1$, construct a countable compact subset K_α of $\mathcal{C}$, whose Cantor–Bendixson rank is exactly α.

11. (a) Let X be a nonempty zero-dimensional Polish space such that all of its compact subsets have empty interior. Fix a complete compatible metric and prove that there

[1] A partition $\mathcal{P}$ of a set X is *trivial* if $\mathcal{P} = \{X\}$.

is a Luzin scheme $(A_s)_{s\in\mathbb{N}^{<\mathbb{N}}}$ of vanishing diameter and satisfying the following properties:

(i) $A_\emptyset = X$;

(ii) A_s is nonempty clopen;

(iii) $A_s = \bigcup_{i\in\mathbb{N}} A_{s^\frown i}$.

Hint: Assuming A_s is defined, cover it by countably many clopen sets of diameter at most $\delta < 1/n$, and choose the δ small enough so that any such cover is necessarily infinite.

(b) Derive the Alexandrov–Urysohn theorem, i.e. show that the Baire space is the only topological space, up to homeomorphism, that satisfies the hypothesis of (a).