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Math 595: Descriptive Set Theory HOMEWORK 1 **Due: the week of Sep 19**

1. Let X be a metric space.

(a) Show that X is complete if and only if every decreasing sequence of closed sets $(B_n)_{n\in\mathbb{N}}$ with $\operatorname{diam}(B_n) \to 0$ has nonempty intersection (in fact, $\bigcap_{n\in\mathbb{N}} B_n$ is a singleton).

(b) Show that the requirement in (a) that $\operatorname{diam}(B_n) \to 0$ cannot be dropped. Do this by constructing a complete metric space that has a decreasing sequence $(B_n)_{n\in\mathbb{N}}$ of closed **balls** with $\bigcap_{n\in\mathbb{N}} B_n = \emptyset$.

HINT: Use $\mathbb{N}$ as the underlying set for your metric space.

2. Let X be a second-countable topological space.

(a) Show that X has at most continuum many open subsets.

HINT: Encode each open set as an element of $\mathscr{P}(\mathbb{N})$.

(b) Let α, β, γ denote ordinals[1]. A sequence of sets $(A_\alpha)_{\alpha<\gamma}$ is called *monotone* if it is either increasing (i.e. $\alpha < \beta \Rightarrow A_\alpha \subseteq A_\beta$, for all $\alpha, \beta < \gamma$) or decreasing (i.e. $\alpha < \beta \Rightarrow A_\alpha \supseteq A_\beta$, for all $\alpha, \beta < \gamma$); call it *strictly monotone*, if all of the inclusions are strict.

Prove that any strictly monotone sequence $(U_\alpha)_{\alpha<\gamma}$ of open subsets of X has countable length, i.e. γ is countable.

HINT: Use the same idea as in the proof of (a).

(c) Show that every monotone sequence $(U_\alpha)_{\alpha<\omega_1}$ open subsets of X eventually stabilizes, i.e. there is $\gamma < \omega_1$ such that for all $\alpha < \omega_1$ with $\alpha \geqslant \gamma$, we have $U_\alpha = U_\gamma$.

HINT: Use the regularity of ω_1. [2]

(d) Conclude that parts (a), (b) and (c) are also true for closed sets.

[1]Here is all you need to know about ordinals. (1) The class of all ordinals is well-ordered under the binary relation $\in$. This means that (1.a) for any two distinct ordinals, either $\alpha \in \beta$ or $\beta \in \alpha$. Moreover, (1.b) if P is a property of ordinals such that at least one ordinal satisfies it, then there is the $\in$-least ordinal satisfying P. A bonus property of ordinals is that $\alpha \in \beta$ implies $\alpha \subseteq \beta$. Thus, for any two ordinals, $\alpha \subseteq \beta$ or $\beta \subseteq \alpha$. In particular, each ordinal itself is a well-ordered set under $\in$. We simply write $<$ instead of $\in$ when comparing two ordinals (in particular, $\{\alpha : \alpha < \beta\} = \beta$). (2) Each well-order is isomorphic to an ordinal. Ordinals are just convenient well-orders to work with, that's all. The first infinite ordinal is denoted by ω or ω_0, and it is countable: $|\omega| = \mathbb{N}$. The first uncountable ordinal is denoted by ω_1. The Continuum Hypothesis says that $|\mathbb{R}| = \omega_1$ but as you know, this is independent of ZFC and we don't assume it.

[2]Instead of defining *regularity of a cardinal* in general, just use the fact that ω_1 cannot be approached by countably many ordinals less than it, i.e. there is no countable increasing sequence of countable ordinals, whose limit/union equals ω_1. This is simply because a countable union of countable sets is countable.

3. Do the annoying work of completing the proof that countable products of Polish spaces are Polish. You have to show that the constructed metric generates the product topology and that it is complete.

   Hint: When showing that the metric-open sets are open in the product topology, it is enough (why?) to show that for each ball B with center x there is an open rectangle containing x that is contained in the ball.

4. By definition, the class of G_δ sets is closed under countable intersections. Show that it is also closed under finite unions. Equivalently, the class of F_σ sets is closed under finite intersections.

   Hint: Put on the logician's hat and think in terms of quantifiers $\forall$ and $\exists$ rather than intersections and unions; for example, if $A = \bigcap_n U_n$, then $x \in A \iff \forall n\, (x \in A_n)$.

5. (a) Show that the Cantor set (with relative topology of $\mathbb{R}$) is homeomorphic to the Cantor space.

   (b) Show that the Baire space $\mathcal{N}$ is homeomorphic to a G_δ subset of the Cantor space $\mathcal{C}$.

   (c) [*Optional*] Show that the set of irrationals (with the relative topology of $\mathbb{R}$) is homeomorphic to the Baire space.

   Hint: Use the continued fraction expansion.

6. Prove that the injection $\iota : X \hookrightarrow 2^{\mathbb{N}}$ given in the coding lemma is such that the ι-preimages of open sets are F_σ.

7. Prove that every Polish space X admits a linear ordering $<$ that is both G_δ and F_σ as a subset of X^2.

   Hint: Binary coding.

8. Let $T \subseteq \mathbb{N}^{<\mathbb{N}}$ be a tree. Define a total ordering $<$ on T such that $<$ is a well-ordering if and only if T doesn't have an infinite branch.

9. [*Optional*] Let S, T be trees on sets A, B, respectively. Prove that for any G_δ set $D \subseteq [S]$ and continuous function $f : D \to [T]$ there is a monotone map $\varphi : S \to T$ such that $f = \varphi^*$. In particular, $\mathrm{dom}(f) = \mathrm{dom}(\varphi^*)$.

   Hint: First solve for $D := [S]$: for $s \in S$, define $\varphi(s)$ to be the longest $t \in T$ such that $|t| \leqslant |s|$ and $N_t \supseteq f(N_s)$. For the general case, write $D = \bigcap\!\!\!\downarrow_n U_n$, where the U_n are open, and replace $|s|$ with the largest $n \leqslant |s|$ such that $N_s \cap D \subseteq U_n$. The case $N_s \cap D = \emptyset$ needs a special (yet straightforward) care.