Math 595: Topics on CBERs HOMEWORK 3 **Due: Apr 11, 5–6:15pm, in 443AH**

1. A graph G on a standard Borel space is called *hyperfinite* if it is a countable increasing union of Borel graphs with all connected components being finite. Prove:

(a) A graph G is hyperfinite if and only if its connectedness relation E_G is hyperfinite.

(b) Any Borel graph with each vertex having degree $\leqslant 2$ is hyperfinite.

2. Retraction along a CBER. Let E, F be countable Borel equivalence relations on standard Borel spaces X, Y, respectively, and let $A \subseteq X$ be a Borel E*-complete section*, i.e., it meets every E-class.

(a) Construct a Borel *retraction to A along E*, i.e., a surjective Borel reduction $\pi : E \twoheadrightarrow E|_A$ whose graph is contained in E, i.e., $\pi(x) E x$ for each $x \in X$.

HINT: Luzin–Novikov (what else).

(b) Deduce that any Borel reduction $f_A : E|_A \to F$ extends to a Borel reduction $f : E \to F$.

3. Take a break with the best of Armenian jazz: Tigran Hamasyan, say, "What the waves brought" or "Revolving - Prayer".

4. For a relation $\prec$ between two CBERs (e.g., $\subseteq$, $\leqslant_B$), we say that a class $\mathcal{E}$ of CBERs is *closed downward* (resp., *upward*) *under* $\prec$ if for any pair E, F of CBERs, $F \in \mathcal{E}$ and $E \prec F$ (resp., $F \prec E$) implies $E \in \mathcal{E}$. Prove that the class of smooth and the class of hyperfinite CBERs are closed downward under $\subseteq$. Are any of these classes closed upward under $\subseteq$?

5. For an equivalence relation E on a set X, a map $f : X \to Y$ is said to be *class-injective* if its restriction to every E-class is injective. For CBERs E, F on standard Borel spaces X, Y, we write $E \to_B^{ci} Y$ if there is a class-injective Borel homomorphism from E to F. Prove that the class of smooth and the class of hyperfinite CBERs are closed downward under $\to_B^{ci}$. Are any of these classes closed upward under $\to_B^{ci}$?

6. For equivalence relations E and F on sets X and Y, define their *product equivalence relation* $E \times F$ on $X \times Y$ by

$$(x, y)\, E \times F\, (x', y') :\Leftrightarrow x\, E\, x' \text{ and } y\, F\, y'.$$

Show that the class of smooth and the class of hyperfinite equivalence relations are closed under products.

7. Maybe some Avishai Cohen (say, "Chutzpan" or "Nu nu") or Esbjorn Svensson Trio (say, "Strange place for snow") or — a convex combination thereof — Phronesis (say, "Zieding").

8. [1](Optional) Let E, F be a CBERs on standard Borel spaces X, Y. Call a Borel set $B \subseteq X$ *E-smooth* if $E|_B$ is smooth. Call a Borel homomorphism $f\colon E \to F$ *smooth* (resp., *smooth-to-one*) if the preimage of every F-class (resp., point in Y) is E-smooth.

(a) Observe that f is smooth if and only if it is smooth-to-one.

(b) Prove that if f is smooth-to-one, then $E \cap \ker f$ is smooth, where $\ker f \coloneqq f^{-1}(\mathrm{Id}_Y)$.

Hint: Use the dichotomy.

(c) Prove that $f = h \circ g$, where $g\colon E \twoheadrightarrow E_A$ is a Borel retraction to some Borel E-complete section $A \subseteq X$ along E, and $h\colon E \to F$ is a class-injective Borel homomorphism.

(d) Conclude that the class of smooth and the class of hyperfinite equivalence relations are closed downward under smooth homomorphisms.

9. This question provides an example (due to Adams, I think) of a Borel 2-regular acyclic graph (a bunch of bi-infinite lines) that does not admit a Borel *directing*, i.e., is not a Cayley graph of any Borel action of $\mathbb{Z}$.

Let $\sigma\colon 2^{\mathbb{N}} \to 2^{\mathbb{N}}$ be the *conjugation* (bit flip) map, i.e., $x \mapsto \overline{x} \coloneqq (\overline{x_n})_n$, where $\overline{b} \coloneqq 1 - b$ for each $b \in \{0,1\}$. Let E_σ be the induced equivalence relation on $2^{\mathbb{N}}$ (each class has two elements) and let $\mathbb{E}_0' \coloneqq \mathbb{E}_0 \vee E_\sigma$, where $\mathbb{E}_0$ is $\mathbb{E}_0$ and $\vee$ is the join[2].

(a) Show that $[\mathbb{E}_0' : \mathbb{E}_0] = 2$.

(b) Let $X \coloneqq 2^{\mathbb{N}} \setminus \left\{x \in 2^{\mathbb{N}} : x \text{ is eventually constant}\right\}$ and recall the odometer action $\mathbb{Z} \curvearrowright X$, where 1 acts as $T : X \to X$. Show that $\sigma \circ T = T^{-1} \circ \sigma$.

(c) Realize what part (b) says in terms of graphs as follows. Let $\vec{G}_T$ be the graph of T, so the directed graph $\vec{H}_T \coloneqq \vec{G}_T \cup \mathrm{Graph}(\sigma)$ looks like a bunch of ladders whose sides are directed lines. Part (b) says that the two sides of each ladder go different direction.

(d) Observe that although $\vec{G}_T$ is asymmetric, its image G_T' under the quotient map $X \twoheadrightarrow X/E_\sigma$ is a symmetric (undirected) 2-regular acyclic graphing of $\mathbb{E}_0'$.

(e) Let μ be the Haar measure on $2^{\mathbb{N}}$ (fair coin-flip). Show that there is no μ-measurable or Baire measurable action of $\mathbb{Z} \curvearrowright X/E_\sigma$ whose standard Cayley graph coincides with G_T'.

Hint: Contradict the ergodicity of $\mathbb{E}_0$ by realizing that any measurable directing of G_T' would choose exactly one side of each ladder (connected component) of $\vec{H}_T$, thus isolating an $\mathbb{E}_0$-invariant subset that's "half" of X.

10. Reward yourself with some Nik Bärtsch, say, "Modul 5" or "Modul 58".[3]

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[1]Thanks to Ronnie for suggesting this.

[2]The *join* $E \vee F$ of equivalence relations E, F on the same set is the equivalence relation generated by $E \cup F$.

[3]Sorry for Spotify links, can't find these album versions elsewhere.