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Math 595: Topics on CBERs Homework 1 **Due: to be determined**

1. Prove that every Polish space X admits a Borel (as a subset of X^2) linear order. In fact, show that there is a linear order that is both F_σ and G_δ.

Hint: Use second-countability.

2. Let E be a finite Borel equivalence relation[1] on a Polish space X.

(a) Show that for each $p \in \mathbb{N}^+$, the set $X_p := \{x \in X : |[x]_E| = p\}$ is Borel. Pinpoint each use of Luzin–Novikov.

(b) Prove that E admits a Borel *transversal*, i.e., a Borel set $S \subseteq X$ that meets every E-class in exactly one point.

Hint: Use Problem 1 and Luzin–Novikov.

(c) Deduce that E-admits a Borel *selector*, i.e., an E-invariant[2] Borel function $s : X \to X$ with $x \, E \, s(x)$ for each $x \in X$.

(d) Take a break and listen to Chopin's Mazurka No.13 in A Minor Op.17 No.4 (performed by Vladimir Ashkenazy).

(e) Now show that E is induced by a Borel automorphism $T : X \to X$ (as an action $\mathbb{Z} \curvearrowright X$), i.e., for each $x \in X$, $[x]_E = \{T^n x : n < |[x]_E|\}$.

Hint: Do this for each X_p separately.

3. Let μ be the $(\frac{1}{2}, \frac{1}{2})$ coin flip measure on $2^{\mathbb{N}}$. Using the 99% lemma for this measure, prove that the equivalence relation E_0 (eventual equality of the sequences) is μ-ergodic.

4. (Feldman–Moore) For a set X, we refer to an pair $(x,y) \in X^2$ as a *directed edge* with *source* x and *target* y. We say that edges (x,y) and $-(x,y) := (y,x)$ are *parallel*. Directed edges $(x,y), (x',y')$ are said to be *source-incident* (resp., *target-incident*, *mixed-incident*) if $x = x'$ (resp., $y = y'$, $x = y'$ or $y = x'$). We say that they are *whatsoever-incident* if they are incident in one of the three aforementioned ways.

Let E be a CBER on a Polish space X.

(a) By the Luzin–Novikov theorem, $G := E \setminus \mathrm{Id}_X = \bigcup_n f_n$, where $f_n : X \rightharpoonup X$ is a Borel partial function. This defines $c_0 : G \to \mathbb{N}$ by $(x,y) \mapsto$ the least $n \in \mathbb{N}$ such that $f_n(x) = y$. Show that c_0 is Borel and that for any distinct source-incident edges $e, e' \in G$, $c_0(e) \neq c(e')$. Hence, for any target-incident edges $e, e' \in G$, $c_0(-e) \neq c_0(-e')$.

(b) Because X is second countable, we can write $X^2 \setminus \mathrm{Id}_X = \bigcup_m (U_m \times V_m)$, where $U_m, V_m \subseteq X$ are open and $U_m \cap V_m = \emptyset$. This defines $c_1 : G \to \mathbb{N}$ by $(x,y) \mapsto$ the least $m \in \mathbb{N}$ such that $(x,y) \in U_m \times V_m$. Show that c_1 is Borel and that for any mixed-incident edges $e, e' \in G$, $c_1(e) \neq c_1(e')$.

[1] This just means each E-class is finite.

[2] For an equivalence relation E on a set X, a function $f : X \to Y$ is *E-invariant* if $x_0 \, E \, x_1 \Rightarrow f(x_0) = f(x_1)$ for all $x_0, x_1 \in X$.

(c) Conclude that $c : G \to \mathbb{N}^3$ defined by $e \mapsto (c_0(e), c_0(-e), c_1(e))$ is a *directed edge-coloring* of G, in the strong sense that any two whatsoever-incident edges get different colors. Thus, G admits a Borel directed edge-coloring with countable-many colors.

(d) Show that for any Borel directed edge-coloring $c : G \to \mathbb{N}$, the map $c' : G \to \mathbb{N}$ defined by $e \mapsto \min\{c(e), c(-e)\}$ is a Borel (*undirected*) *edge-coloring* of G, in the sense that the color of an undirected edge is well-defined (i.e., parallel directed edges get the same color) and incident undirected edges get different colors (i.e., nonparallel whatsoever-incident directed edges get different colors).

5. Take a break and listen to some Bartok, say, Violin Concerto No 2 (has to be Patricia Kopatchinskaja on violin) or Romanian Folk Dances if you prefer something lighter.

6. Let E be a CBER on a standard Borel space X. Prove that E admits a Borel transversal if and only if there is a Borel map $\pi : X \to \mathbb{N}$ such that for each E-class C, $\pi|_C$ is a bijection from C to an initial segment of $\mathbb{N}$.

7. For a set X, call a collection $\mathcal{P}$ of subsets of X a *prepartition of* X if the sets in $\mathcal{P}$ are pairwise disjoint. Denote $\mathrm{dom}(\mathcal{P}) := \bigcup \mathcal{P}$ and call it the *domain* of $\mathcal{P}$.

Now let E be a CBER on a standard Borel space E and let $\mathcal{U}$ be any Borel property of finite nonempty subsets of E-classes, i.e., $\mathcal{U}$ is a Borel subset of $[X]_E^{<\mathbb{N}}$; call the sets in $\mathcal{U}$ *good*. Show that there is a Borel (inclusion) maximal prepartition of X into good sets.

8. Give a full proof that every aperiodic[3] CBER E admits a Borel subequivalence relation $F \subseteq E$ whose each equivalence class has 7 elements.

[3] An equivalence relation is *aperiodic* if every equivalence class is infinite.