Last update: 2:27pm, Apr 17.

Math 432: Set Theory and Topology HOMEWORK 9 **Due: April 18/19**

1. Let (X, d_X) and (Y, d_Y) be metric spaces. Define the following functions $X \times Y \to \mathbb{R}^+$:

$$d_\infty\Big((x_1,y_1),(x_2,y_2)\Big) := \max\{d_X(x_1,x_2), d_Y(y_1,y_2)\}$$
$$d_1\Big((x_1,y_1),(x_2,y_2)\Big) := d_X(x_1,x_2) + d_Y(y_1,y_2)$$
$$d_2\Big((x_1,y_1),(x_2,y_2)\Big) := \sqrt{d_X(x_1,x_2)^2 + d_Y(y_1,y_2)^2}.$$

(a) Prove d_∞ and d_1 are metrics on $X \times Y$.

REMARK: d_2 is also a metric on $X \times Y$, but the proof of the triangle inequality is a bit tedious, so it is left as an *optional* exercise.

(b) Show that these metrics are *equivalent* in the following sense: any pair $d_i, d_j \in \{d_1, d_2, d_\infty\}$ of them admits a constant $C_{ij} > 0$ such that for all $(x,y),(x',y') \in X \times Y$,

$$\frac{1}{C_{ij}} d_i\Big((x,y),(x',y')\Big) \leqslant d_j\Big((x,y),(x',y')\Big) \leqslant C_{ij} d_i\Big((x,y),(x',y')\Big).$$

HINT: It's enough to show for the pairs d_1, d_∞ and d_2, d_∞. (Why?)

(c) Show that the topology on $X \times Y$ defined by d_∞ (equivalently, any of these metrics, but you don't have to show this) is just the product topology. (First, unfold the definitions to see exactly what you have to show.)

2. Define the function $d \colon (\mathbb{N}^\mathbb{N})^2 \to \mathbb{R}^+$ by setting it for distinct $x, y \in \mathbb{N}^\mathbb{N}$ to be $d(x,y) := 2^{-\Delta(x,y)}$, where $\Delta(x,y)$ is the largest $n \in \mathbb{N}$ such that $x|_n = y|_n$, and 0 for $x = y$.

(a) Show that d is an ultrametric on $\mathbb{N}^\mathbb{N}$, i.e., $d(x,z) \leqslant \max\{d(x,y), d(y,z)\}$.

HINT: Draw pictures.

(b) Show that every open ball is of the form $U_s := \Big\{x \in \mathbb{N}^\mathbb{N} : x \supseteq s\Big\}$ for some $s \in \mathbb{N}^{<\mathbb{N}}$.

(c) Show that the sets U_s as above are *clopen*, i.e., both open and closed.

3. For each of the following, determine the boundary and closure of the set A in the metric space (X, d) where $X \subseteq \mathbb{R}$ given below and d is the standard metric on R. Prove your answers.

(a) $X := \mathbb{R}$, $A := \Big\{q \in \mathbb{Q} : q^2 \geqslant 2\Big\}$.

(b) $X := \mathbb{R}$, $A := \Big\{\frac{1}{n} : n \in \mathbb{N} \setminus \{0\}\Big\}$.

(c) $X := [-1,1) \cup \{2\}$, $A := (-1,1)$.

4. Call a set Q in topological space X dense if it intersects every nonempty open set. Prove:

(a) A set Q in X is dense if and only if $\overline{Q} = X$.

(b) $\mathbb{Q}$ is dense in $\mathbb{R}$ (with the standard topology).

(c) The set Q of eventually 0 sequences in $\mathbb{N}^\mathbb{N}$, i.e. $Q := \Big\{x \in \mathbb{N}^\mathbb{N} : \forall^\infty n \in \mathbb{N}\ x(n) = 0\Big\}$, is dense in $\mathbb{N}^\mathbb{N}$. Here $\forall^\infty n$ stands for $\exists m \forall n \geqslant m$.

5. Consider $\mathbb{R}$ with its standard metric.

(a) Show that every open set is a union of open intervals with rational endpoints.

(b) What is the cardinality of the set $\mathcal{U}$ of all open intervals with rational endpoints?

(c) How many open sets are there in $\mathbb{R}$? More precisely, letting $\mathcal{T}$ denote the topology of $\mathbb{R}$, i.e., the set of all open subsets of $\mathbb{R}$, show that $\mathcal{T} \equiv \mathbb{R}$.

HINT: Define a surjection $\mathscr{P}(\mathcal{U}) \twoheadrightarrow \mathcal{T}$.