Math 432: Set Theory and Topology HOMEWORK 3 **Due: Feb 21 (Thu)**

Other (mandatory) **exercises.**

1. Let $<$ be a strict partial order on a set X and let $Y \subseteq X$. Call $y_0 \in Y$ *$<$-minimal in Y* if for any $y \in Y$, $y \not< y_0$. Call $y_1 \in Y$ *$<$-least* (or *$<$-minimum*) *in Y* if for any $y \in Y$, $y_1 \leqslant y$.

(a) Prove that if $y_0 \in Y$ is $<$-least then it is $<$-minimal.

(b) Give an example of $(X,<)$ and $Y \subseteq X$ such that Y admits a $<$-minimal element, but does not admit a $<$-least element.

(c) Give an example of $(X,<)$ and $Y \subseteq X$ such that Y does not admit any $<$-minimal elements.

(d) Prove that if $<$ is a total order, then the converse of part (a) holds: if $y_0 \in Y$ is $<$-minimal, then it is $<$-least.

(e) Conclude that if $<$ is total, then every $Y \subseteq X$ admits at most one $<$-minimal element.

2. Determine which pairs of sets are isomorphic as ordered sets with their usual ordering $<$. Prove your answers.

(a) $\mathbb{N}$ and $\left\{-\frac{1}{n} : n \in \mathbb{N} - \{0\}\right\}$

(b) $\mathbb{Z}$ and $\left\{\frac{1}{n} : n \in \mathbb{Z} - \{0\}\right\} \cup \{0\}$

(c) $\mathbb{R}$ and $(0,1)$

(d) $\mathbb{Q}$ and $[0,1) \cap \mathbb{Q}$

(e) $(0,2)$ and $(0,1) \cup (1,2)$

(f) $(0,2)$ and $(0,1) \cup [2,3)$.

3. (a) Let $(A,<)$ be a well-ordering and let $f : A \to A$ be an *order-homomorphism*, i.e.

$$a_0 < a_1 \implies f(a_0) < f(a_1)$$

for all $a_0, a_1 \in A$. Prove that f *progressive*, i.e., $a \leqslant f(a)$ for all $a \in A$.

(b) Deduce directly from part (a) that $(A,<) \not\cong (A,<)$ for any well-ordering $(A,<)$.

Remark. We proved this statement in class as a corollary of the uniqueness lemma for isomorphisms witnessing $\preceq$. The purpose of this exercise is to give a more direct proof.

(c) Ordering $\mathbb{N}^2$ lexicographically, give an example of an order-homomorphism $f : \mathbb{N}^2 \to \mathbb{N}^2$ (other than the identity map) such that $f(n,m) = (n,m)$ for all $(n,m) \geqslant_{\text{lex}} (2,0)$.

4. Let $(A,<)$ and $(B,<)$ be well orderings.

(a) Prove that there is a set F such that

$$F = \{f : f \text{ is an order isomorphism between initial segments of } (A,<) \text{ and } (B,<)\}.$$

(b) Prove that for any $f, g \in F$, $f \subseteq g$ or $g \subseteq f$.

(c) Conclude that $f := \bigcup F$ is an order isomorphism (in particular, a function) of an initial segment A' of $(A, <)$ with an initial segment of B' of $(B, <)$.

(d) Prove that $A' = A$ or $B' = B$.