Math 574: DST HOMEWORK 4 **Due: Nov 5 in problem session**

1. Let $X = \mathbb{I}^{\mathbb{N}}$ and put $C_0 = \{(x_n)_{n\in\mathbb{N}} : x_n \to 0\}$. Show that C_0 is in $\mathbf{\Pi}^0_3(X)$.

2. A class $\mathbf{\Gamma}$ of sets is called *self-dual* if it is closed under complements, i.e. $\neg\mathbf{\Gamma} = \mathbf{\Gamma}$. Show that if $\mathbf{\Gamma}$ is a self-dual class of sets in topological spaces that is closed under continuous preimages, then for any topological space X there does not exist an X-universal set for $\mathbf{\Gamma}(X)$. Conclude that neither the class $\mathcal{B}(X)$ of Borel sets, nor the classes $\mathbf{\Delta}^0_\xi(X)$, can have X-universal sets.

3. Letting X be a separable metrizable space and $\lambda < \omega_1$ be a limit ordinal, put

$$\mathbf{\Omega}^0_\lambda(X) := \bigcup_{\xi<\lambda} \mathbf{\Sigma}^0_\xi(X) \ (= \bigcup_{\xi<\lambda} \mathbf{\Delta}^0_\xi(X) = \bigcup_{\xi<\lambda} \mathbf{\Pi}^0_\xi(X)).$$

(a) Let Y be an uncountable Polish space and prove that there exists a set $P \in \mathbf{\Delta}^0_\lambda(Y \times X)$ that parameterizes $\mathbf{\Omega}^0_\lambda(X)$.

HINT: First construct such a set for $Y = \mathbb{N} \times \mathcal{C}$. Then conclude it for $Y = \mathcal{C}$ using the fact that the following functions are continuous: $()_0 : \mathcal{C} \to \mathbb{N}$ and $()_1 : \mathcal{C} \to \mathcal{C}$ defined for $y \in \mathcal{C}$ by

$$y = 1^{(y)_0} {}^\frown 0 {}^\frown (y)_1.$$

Finally, conclude the statement for any Y using the perfect set property.

(b) Conclude that if X is uncountable Polish, then $\mathbf{\Delta}^0_\lambda(X) \supsetneq \mathbf{\Omega}^0_\lambda(X)$.

4. (a) Show that any Polish space admits a finer Polish topology that is zero-dimensional and has the same Borel sets, i.e. for a given Polish space $(X, \mathcal{T})$, there exists a zero-dimensional Polish topology $\mathcal{T}_0 \supseteq \mathcal{T}$ such that $\mathcal{B}(\mathcal{T}_0) = \mathcal{B}(\mathcal{T})$.

(b) Let G be a countable group and consider a Borel action of G on a Polish space $(X, \mathcal{T})$, i.e. each $g \in G$ acts as a Borel automorphism of X. Prove that there exists a Polish topology $\mathcal{T}_0 \supseteq \mathcal{T}$ with $\mathcal{B}(\mathcal{T}_0) = \mathcal{B}(\mathcal{T})$ that makes the action of G continuous. Moreover, $\mathcal{T}_0$ can be taken to be zero-dimensional.

5. Let X, Y be topological spaces and let $\text{proj}_X : X \times Y \to X$ be the projection function. Prove the following statements:

(a) proj_X is continuous and open.

(b) proj_X does not in general map closed sets to closed sets, even for $X = Y = \mathbb{R}$.

REMARK: We will see shortly in the course that for certain $Y = \mathcal{N}$, the projection of a closed set may not even be Borel in general.

(c) For $X = Y = \mathbb{R}$, proj_X maps closed sets to σ-compact (and hence F_σ) sets. More generally, images of F_σ sets under continuous functions from σ-compact to Hausdorff spaces are F_σ.

(d) **Tube lemma:** If Y is compact, then proj_X indeed maps closed sets to closed sets.

Hint: It is perhaps tempting to use sequences, but this would only work for first-countable spaces. Instead, use the open cover definition of compact and show that for closed $F \subseteq X \times Y$, every point $x \in X \setminus \text{proj}_X(F)$ has an open neighborhood disjoint from $\text{proj}_X(F)$. The "correct" solution should use nothing but definitions.

6. Let X be Polish and let E be an analytic equivalence relation on X, i.e. E is an analytic subset of X^2.

(a) Show that for an analytic set A, its saturation $[A]_E := \{x \in X : \exists y \in A (x \, E \, y)\}$ is also analytic.

(b) Let $A, B \subseteq X$ be disjoint E-invariant analytic sets (i.e., $[A]_E = A$, $[B]_E = B$). Prove that there is an E-invariant Borel set D separating A and B, i.e., $D \supseteq A$ and $D \cap B = \emptyset$.