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Math 574: DST HOMEWORK 3 **Due: Oct 22 in problem session**

1. Prove that a topological group G is Baire iff G is nonmeager.

2. Let G be a Polish group (i.e. a topological group whose topology happens to be Polish) and let $H < G$ be a subgroup. Prove that H is Polish iff H is closed.

HINT: Consider H inside $\overline{H}$. What is the Baire category status (meager/nonmeager/comeager) of H in (the relative topology of) $\overline{H}$? If $H \subsetneq \overline{H}$, look at the cosets.

3. Let Γ be a group acting on a Polish space X by homeomorphisms (i.e. each element $\gamma \in \Gamma$ acts as a homeomorphism of X). A set $A \subseteq X$ is called invariant if $\gamma A = A$ for all $\gamma \in \Gamma$. The action $\Gamma \curvearrowright X$ is called *generically ergodic* if every invariant Baire measurable set $A \subseteq X$ is either meager or comeager. For a set $A \subseteq X$, denote by $[A]_\Gamma$ the saturation of A, namely $[A]_\Gamma = \bigcup_{\gamma \in \Gamma} \gamma A$.

Prove that the following are equivalent:

(1) $\Gamma \curvearrowright X$ is generically ergodic.

(2) Every invariant nonempty open set is dense.

(3) For comeager-many $x \in X$, the orbit $[x]_\Gamma$ is dense.

(4) There is a dense orbit.

(5) For every nonempty open sets $U, V \subseteq X$, there is $\gamma \in \Gamma$ such that $(\gamma U) \cap V \neq \emptyset$.

HINT: For (2)$\Rightarrow$(3), take a countable basis $\{U_n\}_{n \in \mathbb{N}}$ and consider $\bigcap_n [U_n]_\Gamma$.

4. **(If you're curious)**[1] Show that the Kuratowski–Ulam theorem fails if A is not Baire measurable by constructing a nonmeager set $A \subseteq \mathbb{R}^2$ (using AC) so that no three points of A are on a straight line.

HINT: Note that there are only continuum many F_σ sets, so take a transfinite enumeration $(F_\xi)_{\xi < 2^{\aleph_0}}$ of all *meager* F_σ sets, and construct a sequence $(a_\xi)_{\xi < 2^{\aleph_0}}$ of points in $\mathbb{R}^2$ by transfinite recursion so that for each $\xi < 2^{\aleph_0}$,

$$\{a_\lambda : \lambda \leqslant \xi\} \not\subseteq F_\xi,$$

and no three of the points in $\{a_\lambda : \lambda \leqslant \xi\}$ lie on a straight line.

HINT: Recall that in perfect Polish spaces (such as $\mathbb{R}, \mathbb{R}^2$), any nonmeager Baire measurable subset contains a copy of the Cantor space (this is because it contains a nonmeager G_δ set). Now if $A := \{a_\lambda : \lambda \leqslant \xi\} \subseteq F_\xi$, apply Kuratowski–Ulam to F_ξ to find $x \in \mathbb{R}$ such that $(F_\xi)_x$ is meager and the vertical line $L_x = \{(x, y) \in \mathbb{R} : y \in \mathbb{R}\}$ is disjoint from A.

5. Using the outline below, prove Pettis's theorem:

[1] Problems marked with this are interesting, but not essential for understanding the course material.

Theorem (Pettis)**.** *Let G be a topological group and $A \subseteq G$ be Baire measurable. If A is nonmeager, then $A^{-1}A$ contains an open neighborhood of the identity 1_G; in fact if $U \Vdash A$, then $U^{-1}U \subseteq A^{-1}A$.*

1) By Question 1, G must be Baire.
2) Note that for any sets $B, C \subseteq G$,
$$B \subseteq C^{-1}C \iff \forall h \in B\, (Ch \cap C \neq \emptyset). \qquad (*)$$
3) Let $U \subseteq G$ be nonempty open such that $U \Vdash A$. Fix arbitrary $g \in U$ and note that $V := g^{-1}U \subseteq U^{-1}U$ is an open neighborhood of 1_G. Thus, by $(*)$, $\forall h \in V$, $Uh \cap U \neq \emptyset$.
4) Conclude that for each $h \in V$, $Ah \cap A \neq \emptyset$, and hence, by $(*)$ again, $V \subseteq A^{-1}A$.
5) Note that we have shown $g^{-1}U \subseteq A^{-1}A$ for arbitrary $g \in U$, and thus, $U^{-1}U \subseteq A^{-1}A$.

6. Let G be a Baire topological group (i.e. G is nonmeager) and let $H < G$ be a Baire measurable subgroup. Prove if H is nonmeager then it is actually clopen! In particular, if H has countable index in G, then it is clopen.

7. (a) **Automatic continuity:** Let G, H be topological groups, where G is Baire and H is separable. Then every Baire measurable group homomorphism $\varphi : G \to H$ is actually continuous!

Hint: Enough to prove continuity at 1_G, so let $U \ni 1_H$ be open and take an open neighborhood $V \ni 1_H$ such that $V^{-1}V \subseteq U$. Using the separability of H, deduce that $\varphi^{-1}(hV)$ is nonmeager for some $h \in H$ and apply Pettis's theorem.

(b) Conclude that if $f : (\mathbb{R}, +) \to (\mathbb{R}, +)$ is a Baire measurable group homomorphism, then for some $a \in \mathbb{R}$, $f(x) = ax$ for all $x \in \mathbb{R}$.

Hint: First show this for integers, then for rationals, etc.

8. **(If you're curious)** Letting $d_A : \mathbb{R} \to [0,1]$ denote the Lebesgue density function for $A \subseteq \mathbb{R}$, define $D(A) := \{x \in \mathbb{R} : d_A(x) = 1\}$. Show that Lebesgue measurable subsets $A \subseteq \mathbb{R}$ with $A \subseteq D(A)$ form a topology.

Hint: To show that a possibly uncountable union $A := \bigcup_{\ell \in L} A_\ell$ of such sets is still Lebesgue measurable, prove that it can be approximated from above by open sets. First reduce it to the case $A \subseteq (0,1)$. Then for each $\varepsilon > 0$, let $\mathcal{C}$ be the collection of all intervals $I \subseteq (0,1)$ that admit an $\ell \in L$ with $\frac{\lambda(A_\ell \cap I)}{\lambda(I)} > 1 - \varepsilon$. Show that $\mathcal{C}$ covers A and, in fact, there is a subcover $\mathcal{C}' \subseteq \mathcal{C}$ of A with $\lambda(\bigcup \mathcal{C}' \setminus A) < 2\varepsilon$. For the latter, use the version of the Vitali covering lemma as in Lemma 8 of the author's note on Lebesgue differentiation.

9. Prove the following facts about the density topology on $\mathbb{R}$. (Below λ denotes the Lebesgue measure on $\mathbb{R}$ and all topological terms are with respect to the density topology.)

(a) Every nonempty open set has positive measure.

(b) For a Lebesgue measurable set $A \subseteq \mathbb{R}$, explicitly compute $\operatorname{Int}(A)$ and $\overline{A}$, and conclude that $\lambda(\operatorname{Int}(A)) = \lambda(A) = \lambda(\overline{A})$.

10. Consider $\mathbb{R}$ with the density topology and Lebesgue measure λ. For $A \subseteq \mathbb{R}$, prove that the following are equivalent:

(1) A is nowhere dense in the density topology;

(2) A is meager in the density topology;

(3) A is λ-null.

Conclude that A is Baire measurable in the density topology if and only if it is Lebesgue measurable.