Last update: 12:48 pm, Oct 1.

Math 574: DST HOMEWORK 2 **Due: Oct 3 in problem session**

1. Let (X,d) be a metric with $d \leqslant 1$. For $(K_n)_n \subseteq \mathcal{K}(X) \setminus \{\emptyset\}$ and nonempty $K \in \mathcal{K}(X)$:

(a) $\delta(K, K_n) \to 0 \Rightarrow K \subseteq \underline{\mathrm{T}\lim}_n K_n$;

(b) $\delta(K_n, K) \to 0 \Rightarrow K \supseteq \overline{\mathrm{T}\lim}_n K_n$.

In particular, $d_H(K_n, K) \to 0 \Rightarrow K = \mathrm{T}\lim_n K_n$. Show that the converse may fail.

Op1. (Optional) Let X be metrizable.

(a) The relation "$x \in K$" is closed, i.e. $\{(x,K) : x \in K\}$ is closed in $X \times \mathcal{K}(X)$.

(b) The relation "$K \subseteq L$" is closed, i.e. $\{(K,L) : K \subseteq L\}$ is closed in $\mathcal{K}(X)^2$.

(c) The map $(K,L) \mapsto K \cup L$ from $\mathcal{K}(X)^2$ to $\mathcal{K}(X)$ is continuous.

(d) Find a compact X for which the map $(K,L) \mapsto K \cap L$ from $\mathcal{K}(X)^2$ to $\mathcal{K}(X)$ is not continuous.

2. Let X be a nonempty perfect Polish space and let Q be a countable dense subset of X. Show that Q is F_σ but not G_δ. In particular, $\mathbb{Q}$ is not Polish (in the relative topology of $\mathbb{R}$).

Op2.[1] **(Optional)** Show that $[0,1]$ does not admit a countable nontrivial[2] partition into closed intervals.

HINT: What kind of subset would the endpoints of those intervals form?

3. A *topological group* is a group with a topology on it so that group multiplication $(x,y) \to xy$ and inverse $x \to x^{-1}$ are continuous functions. Show that a countable topological group is Polish if and only if it is discrete.

4. Let X be separable metrizable and let

$$\mathcal{K}_p(X) := \{K \in \mathcal{K}(X) : K \text{ is perfect}\}.$$

(a) Show that $\mathcal{K}_p(X)$ is a G_δ set in $\mathcal{K}(X)$. In particular, if X is Polish, then so is $\mathcal{K}_p(X)$.

(b) Show that if X is nonempty perfect Polish, then $\mathcal{K}_p(X)$ is dense in $\mathcal{K}(X)$. Conclude that a generic compact subset of X is perfect.

5. (a) Let X be a nonempty zero-dimensional Polish space such that all of its compact subsets have empty interior. Fix a complete compatible metric and prove that there is a Luzin scheme $(A_s)_{s \in \mathbb{N}^{<\mathbb{N}}}$ with vanishing diameter and satisfying the following properties:

[1] Thanks to Jenna Zomback for sparking this problem.

[2] A partition $\mathcal{P}$ of a set X is *trivial* if $\mathcal{P} = \{X\}$.

(i) $A_\emptyset = X$;

(ii) A_s is nonempty clopen;

(iii) $A_s = \bigcup_{i \in \mathbb{N}} A_{s^\frown i}$.

Hint: Assuming A_s is defined, cover it by countably many clopen sets of diameter at most $\delta < 1/n$, and choose the δ small enough so that any such cover is necessarily infinite.

(b) Derive the Alexandrov–Urysohn theorem, i.e. show that the Baire space is the only topological space, up to homeomorphism, that satisfies the hypothesis of (a).

6. Let $Y \subseteq \mathbb{R}$ be G_δ and such that Y, $\mathbb{R} \setminus Y$ are dense in $\mathbb{R}$. Show that Y is homeomorphic to $\mathbb{N}^\mathbb{N}$. In particular, $\mathbb{R} \setminus \mathbb{Q}$ is homeomorphic to $\mathbb{N}^\mathbb{N}$.

Op3.[3] **(Optional)**

(a) Give an example of a function $f : \mathbb{R} \to \mathbb{R}$ that is continuous at every irrational but discontinuous at every rational.

(b) Prove that there is no function $f : \mathbb{R} \to \mathbb{R}$ that is continuous at every rational but discontinuous at every irrational.

Hint: Show that the set of continuity points of any function is G_δ.

7. A *finite game* on a set A is a game similar to infinite games, but the players play only finitely many steps before the winner is decided. More formally, it is a (possibly infinite) tree $T \subseteq A^{<\mathbb{N}}$ that has no infinite branches, and the set of runs is $\mathrm{Leaves}(T)$, so the payoff set is a subset $D \subseteq \mathrm{Leaves}(T)$. Player I wins the run $s \in \mathrm{Leaves}(T)$ of the game iff $s \in D$. Consequently, Player II wins iff $s \in \mathrm{Leaves}(T) \setminus D$.

(a) Prove the determinacy of finite games.

Hint: Call a position $s \in T$ determined, if from that point on, one of the players has a winning strategy. Thus, no player has a winning strategy in the beginning iff $\emptyset$ is undetermined. What can you say about extensions of undetermined positions?

(b) Conclude the determinacy of clopen infinite games. (These are games with runs in $A^\mathbb{N}$ and the payoff set a clopen subset of $A^\mathbb{N}$.)

Op4.[4] **(Optional)** Let $\mathcal{G}$ be the so-called *Hamming graph* on $2^\mathbb{N}$, namely, there is an edge between $x, y \in 2^\mathbb{N}$ exactly when x and y differ by one bit.

(a) Prove that $\mathcal{G}$ is has no odd cycles and hence is bipartite (admits a 2-coloring). Pinpoint the use of AC.

[3] Thanks to Francesco Cellarosi for bringing up the statements of this question to me.

[4] Thanks to Forte Shinko for suggesting this problem.

(b) Fix a coloring $c : 2^{\mathbb{N}} \to 2$ of $\mathcal{G}$ and let $A_i \coloneqq c^{-1}(i)$ for $i \in \{0,1\}$. Consider the game where each player plays a finite nonempty binary sequence at each step and a play is the concatenation of those finite sequences, thus an infinite binary sequence. Prove that this game with the payoff set A_0 is not determined by showing that if one of the players had a winning strategy, so would the other one.

Hint: Steal the other player's strategy.