Last update: 9:29pm, Sept 10.

Math 574: DST HOMEWORK 1 **Due: Sep 17 in the problem session**

1. Prove that any separable metric space has cardinality at most continuum.

REMARK: This is true more generally for first-countable separable Hausdorff topological spaces, but false for general separable Hausdorff topological spaces (try to construct a counter-example).

2. Let X be a second-countable topological space.

(a) Show that X has at most continuum-many open subsets.

(b) **(Optional)**[1] Let α, β, γ denote ordinals. A sequence of sets $(A_\alpha)_{\alpha<\gamma}$ is called *monotone* if it is either increasing (i.e. $\alpha < \beta \Rightarrow A_\alpha \subseteq A_\beta$, for all $\alpha, \beta < \gamma$) or decreasing (i.e. $\alpha < \beta \Rightarrow A_\alpha \supseteq A_\beta$, for all $\alpha, \beta < \gamma$); call it *strictly monotone*, if all of the inclusions are strict.

Prove that any strictly monotone sequence $(U_\alpha)_{\alpha<\gamma}$ of open subsets of X has countable length, i.e. γ is countable.

HINT: Use the same idea as in the proof of (a).

(c) **(Optional)** [1] Show that every monotone sequence $(U_\alpha)_{\alpha<\omega_1}$ open subsets of X eventually stabilizes, i.e. there is $\gamma < \omega_1$ such that for all $\alpha < \omega_1$ with $\alpha \geqslant \gamma$, we have $U_\alpha = U_\gamma$.

HINT: Use the regularity of ω_1, i.e. supremum of countably-many countable ordinals is still a countable ordinal.

(d) Conclude that parts (a), (b) and (c) are also true for closed sets.

3. (a) Show that a metric space X is complete if and only if every decreasing sequence of closed sets $(B_n)_{n\in\mathbb{N}}$ with $\mathrm{diam}(B_n) \to 0$ has nonempty intersection (in fact, $\bigcap_{n\in\mathbb{N}} B_n$ is a singleton).

(b) **(Optional)** Show that the requirement in (a) that $\mathrm{diam}(B_n) \to 0$ cannot be dropped. Do this by constructing a complete metric space that has a decreasing sequence $(B_n)_{n\in\mathbb{N}}$ of closed **balls** with $\bigcap_{n\in\mathbb{N}} B_n = \emptyset$.

HINT: Use $\mathbb{N}$ as the underlying set for your metric space.

4. Prove that every Polish space X admits a linear ordering $<$ that is both G_δ and F_σ as a subset of X^2.

HINT:[2] Think of the points of X as binary sequences using a countable basis $(U_n)_{n\in\mathbb{N}}$.

[1] Parts (b) and (c) of Problem 2 are the only questions in the entire course that mention ordinals.

[2] Thanks to Jenna Zomback for this.

5. Let $T \subseteq A^{<\mathbb{N}}$ be a tree and suppose it is finitely branching. Prove (directly) that $[T]$ is sequentially compact.

6. Let $T \subseteq \mathbb{N}^{<\mathbb{N}}$ be a tree. Define a total ordering $<$ on T such that $<$ is a well-ordering if and only if T doesn't have an infinite branch.

7. Let S, T be trees on sets A, B, respectively. Let $D \subseteq [S]$ be a G_δ set and let $f : D \to [T]$ be a continuous function.
   (a) Assuming that $D = [S]$, prove that there is a monotone map $\varphi : S \to T$ such that $f = \varphi^*$.
   Hint: For $s \in S$, define $\varphi(s)$ to be the longest $t \in T$ such that $|t| \leqslant |s|$ and $N_t \supseteq f(N_s)$, where $N_s := \{x \in [S] : x \supseteq s\}$ and $N_t := \{y \in [T] : y \supseteq t\}$.
   (b) **(Optional)** Letting D be an arbitrary G_δ set, prove that there is a monotone map $\varphi : S \to T$ such that $D = \mathrm{dom}(\varphi^*)$ and $f = \varphi^*$.
   Hint: Write $D = \bigcap\!\!\!\downarrow_n U_n$, where the U_n are open, and in the hint above replace $|s|$ with the largest $n \leqslant |s|$ such that $N_s \cap D \subseteq U_n$. The case $N_s \cap D = \emptyset$ needs a special (yet straightforward) care.

8. Let X be a compact metric space and Y be a separable complete metric space. Let $C(X, Y)$ be the space of continuous functions from X to Y equipped with the uniform metric, i.e. for $f, g \in C(X, Y)$,

$$d_u(f, g) = \sup_{x \in X} d_Y(f(x), g(x)).$$

Prove that $C(X, Y)$ is a separable complete metric space, hence Polish.

Hint 1: Proving separability is tricky, so you may want to first prove it for $X = [0, 1]$ and $Y = \mathbb{R}$. In the general case (to prove separability), note that by uniform continuity,

$$C(X, Y) = \bigcup_n A_{n,m},$$

for every $n \in \mathbb{N}$, where

$$A_{n,m} = \{f \in C(X, Y) : \forall x, y \in X (d_X(x, y) < 1/n \Rightarrow d_Y(f(x), f(y)) < 1/m)\}.$$

Realize that it is enough to show that for any $n, m \in \mathbb{N}$, there is a countable $B_{n,m} \subseteq A_{n,m}$ such that for any $f \in A_{n,m}$ there is $g \in B_{n,m}$ with $d_u(f, g) < 3/m$. Now fix n, m and try to construct $B_{n,m}$; when doing so, don't try to *define* each function in $B_{n,m}$ by hand as you would maybe do in the case $X = [0, 1]$; instead, carefully *pick* them out of functions in $A_{n,m}$.

Hint 2: This is Theorem 4.19 in Kechris's "Classical Descriptive Set Theory".

9. Show that Hausdorff metric on $\mathcal{K}(X)$ is compatible with the Vietoris topology.