12:13am, Apr 19.

Math 574: Set Theory HOMEWORK 9 **Due: Apr 19 and 20**

Notation. For sets A, B, let $[A \rightharpoonup B]^{<\omega}$ denote the set of all finite partial functions from A to B (i.e. partial functions with finite domain). We make this set into a poset by taking as $\leqslant$ the relation $\supseteq$ (reverse inclusion).

Below, let M be a countable transitive model of ZF and let $(\mathbb{P}, \leqslant) \in M$ be a poset.

1. Recall that $X_{\mathbb{P}} \subseteq 2^{\mathbb{P}}$ denotes the Polish space of all $\mathbb{P}$-ultrafilters and the sets

$$U_p := \{G \in X_{\mathbb{P}} : p \in G\},\ p \in \mathbb{P},$$

form a basis for the topology on $X_{\mathbb{P}}$. Let $Q \subseteq \mathbb{P}$ and let $\mathcal{U}_Q := \left\{U_p : p \in Q\right\}$. Characterize the following properties of Q in terms of $\mathcal{U}_Q$ and the topology of $X_{\mathbb{P}}$. You may need the following:

Definition. A collection $\mathcal{W}$ of open subsets of a topological space X is called a *weak basis* of X if for each nonempty open $U \subseteq X$ there is a nonempty $W \in \mathcal{W}$ with $W \subseteq U$.

(a) Q is predense $\iff \ldots$

(b) Q is an antichain $\iff \ldots$

(c) Q is a maximal antichain $\iff \ldots$

(d) Assume in addition that for any $p, q \in \mathbb{P}$ with $p \not\leqslant q$, there is $r \leqslant p$ with $r \perp q$.

Q is dense $\iff \ldots$

REMARK: If you think this problem is too easy, you are right. I just want to establish a poset $\leftrightarrow$ topology translation.

2. Let $D \subseteq \mathbb{P}$, $D \in M$, be dense. Prove the following.

(a) Density is transitive: if $D' \subseteq D$ is dense in the poset $(D, \leqslant)$, then D' is dense in $(\mathbb{P}, \leqslant)$.

(b) If $G \subseteq \mathbb{P}$ is an M-generic $\mathbb{P}$-ultrafilter, then $G \cap D$ is an M-generic D-ultrafilter.

(c) Conversely, if $G \subseteq D$ is an M-generic D-ultrafilter, then the upward closure $\overline{G}$ of G in $\mathbb{P}$ is the unique (prove the uniqueness) M-generic $\mathbb{P}$-ultrafilter with $\overline{G} \cap D = G$.

3. For sets $A, B \in M$, let $\mathbb{P} := [A \rightharpoonup B]^{<\omega}$ and recall (prove!) that $\mathbb{P} \in M$. Let $G \subseteq \mathbb{P}$ be an M-generic ultrafilter and put $g := \bigcup G$.

(a) Assuming that A is infinite, prove that g is a surjective (entire) function from A to B.

HINT: For each $b \in B$, define a dense set $D_b \subseteq \mathbb{P}$ in M.

(b) Let $A := S \times \omega$, for some $S \in M$, and let $B := 2$. For each $s \in S$, put $g_s := g(s, \cdot) : \omega \to 2$, i.e. $g_s \in 2^{\omega}$. Thus, g can be viewed as a function $g' : S \to 2^{\omega}$ given by $s \mapsto g_s$. Prove that g' is injective.

4. (General Mostowski collapse) Let R be a binary relation on a set X. Say that R is *well-founded* if for every nonempty $Y \subseteq X$ there is $y \in Y$ with $R^y := \{x \in X : xRy\}$ is disjoint from Y. Prove that this is equivalent to the inexistence of a sequence $(x_n)_{n\in\omega}$ of elements of X with $x_{n+1}Rx_n$ for each $n \in \omega$.

A function π on X is called an *R-collapse* (or an *R-contraction*) if for every $y \in X$, $\pi(y) = \{\pi(x) : x \in X \text{ with } xRy\}$. Following the steps below, prove (in ZF − Pow) that every well-founded binary relation R admits a unique R-collapse.

(i) Uniqueness: Suppose towards a contradiction that there are two distinct such functions and consider the set of points at which they differ.

(ii) Towards existence: Say that $A \subseteq X$ is *downward R-closed* if for each $a \in A$, $R^a \subseteq A$. Let π be the union of all R-collapses defined on downward R-closed subsets of X and show that π itself is an R-contraction on a downward R-closed subset of X.

(iii) Existence: Show that $\mathrm{dom}(\pi) = X$.

5. A set (think: family of sets) $\mathcal{F}$ is called a Δ-*system* if there is a set r (called the *root*) such that for any distinct $a, b \in \mathcal{F}$, $a \cap b = r$. Prove:

Δ-system Lemma. *Every uncountable family $\mathcal{F}$ of finite sets contains an uncountable Δ-system.*

Hint: May assume without loss of generality (why?) that all sets in $\mathcal{F}$ have the same size $n \in \omega$. Prove by induction on n.