Math 574: Set Theory HOMEWORK 8 **Due: Apr 12 and 13**

Terminology. For a filter $\mathcal{F}$ on a set X, say that a set $A \subseteq X$ is *$\mathcal{F}$-large* (resp. *$\mathcal{F}$-small*) if $A \in \mathcal{F}$ (resp. $A \in \mathcal{F}'$, i.e. $A^c \in \mathcal{F}$). For a property P of elements of X (in other words, $P \subseteq X$), say that $\mathcal{F}$-a.e. $x \in X$ has property P, written

$$\forall^{\mathcal{F}} x \in X \ (x \text{ has property } P),$$

if the set $\{x \in X : x \text{ has property } P\}$ is $\mathcal{F}$-large.

1. For a set A, let $[A]^2$ denote the collection of all subsets of A of size exactly 2; think of these as edges between the elements of A. Prove the following using an ultrafilter on $\mathbb{N}$.

Infinite Ramsey Theorem. *For any finite coloring of $[\mathbb{N}]^2$, i.e. a function $c : [\mathbb{N}]^2 \to k := \{0, 1, \dots, k-1\}$, there is an infinite monochromatic set $A \subseteq \mathbb{N}$, i.e. $c|_{[A]^2}$ is constant.*

REMARK: The usage of an ultrafilter here is an overkill, of course, but a nice one.

HINT: Letting μ be an ultrafilter on $\mathbb{N}$, derive a coloring c' of $\mathbb{N}$ as follows: for each vertex $v \in \mathbb{N}$, look at the edges incident to v and take the μ-large color. Having colored $\mathbb{N}$, one of these colors is μ-large. Build a monochromatic subsequence inside this color by induction.

2. Call a set $D \subseteq \mathbb{Z}$ a *difference set* or a *Δ-set* if there is a sequence $(z_n)_{n \in \mathbb{N}} \subseteq \mathbb{Z}$ of pairwise distinct elements such that $D := \{z_n - z_m : n > m\}$. Denote the collection of all Δ-sets by Δ.

(a) Show that Δ-sets have the *Ramsey property*; namely, for any $D \in \Delta$, whenever D is partitioned into two sets, at least one of them contains a Δ-set.

(b) Conclude that $\Delta^* := \{A \subseteq \mathbb{Z} : \forall D \in \Delta \ (A \cap D \neq \emptyset)\}$ is a filter.

3. Let X be a topological space and $\mathcal{F}$ a filter on $\mathbb{N}$. Call $x \in X$ an *$\mathcal{F}$-limit* of a sequence $(x_n)_{n \in \mathbb{N}} \subseteq X$, written

$$\lim_{n \to \mathcal{F}} x_n = x,$$

if for every open neighborhood U of x, $x_n \in U$ for $\mathcal{F}$-a.e. $n \in \mathbb{N}$.

(a) For which filter $\mathcal{F}$, the notion of $\mathcal{F}$-limit coincides with the usual limit?

(b) Now let μ be an ultrafilter on $\mathbb{N}$ and suppose that X is compact (open covers definition). Prove that every sequence in X has a μ-limit.

Definition. Let $(\mathbb{P}, \leqslant)$ be a partially ordered set, which we call a *poset*, for short. Say that $p, q \in \mathbb{P}$ are *compatible* if they admit a *common strengthening*, i.e. there is $r \in \mathbb{P}$ with $r \leqslant p$ and $r \leqslant q$. Otherwise, say that p and q are *incompatible*, written $p \perp q$.

Definition. For a poset $(\mathbb{P}, \leqslant)$, a set $F \subseteq \mathbb{P}$ is called a *$\mathbb{P}$-filter* if

(i) (Upward closed) $p \in F$ and $q \geqslant p \implies q \in F$;

(ii) (Contains common strengthenings) $p, q \in F \implies$ there is $r \in F$ with $r \leqslant p$ and $r \leqslant q$.

Call a $\mathbb{P}$-filter F an *ultrafilter* (or a *strongly maximal filter*) if for each $p \in \mathbb{P}$ either $p \in F$ or there is $q \in F$ with $p \perp q$.

4. Let A, B be sets and let $\mathbb{P}$ be the set of all partial functions $A \rightharpoonup B$ with finite domain. We turn $\mathbb{P}$ into a poset under extension (i.e. reverse inclusion), i.e. for any $p, q \in \mathbb{P}$, $p \leqslant q :\Leftrightarrow p \supseteq q$. Show that the map $F \mapsto \cup F$ is a bijection between the set of ultrafilters and the set of functions $A \to B$.

5. Let $(\mathbb{P}, \leqslant)$ be a countable poset. Identifying $X := \mathscr{P}(\mathbb{P})$ with $2^{\mathbb{P}}$ equips X with the product topology, making it a homeomorphic copy of the Cantor space. Let $X_{\mathbb{P}} \subseteq X$ denote the set of all $\mathbb{P}$-ultrafilters.

(a) Show that $X_{\mathbb{P}}$ is a G_δ subset of X, i.e. a countable intersection of open sets.

Remark: X is a Polish space[1] (being compact metrizable) and a subset of a Polish space is itself a Polish space (in the relative topology) if and only if it is G_δ (see, for example, Proposition 1.7 in my DST notes). Thus, $X_{\mathbb{P}}$ is also a Polish space.

(b) Prove that the sets

$$U_p := \{G \in X_{\mathbb{P}} : p \in G\},\ p \in \mathbb{P},$$

form a basis for the topology on $X_{\mathbb{P}}$.

(c) Express in terms of the U_p what it means for a set $A \subseteq X_{\mathbb{P}}$ to be

(i) open,

(ii) dense.

[1] A topological space is called *Polish* if it is separable and admits a compatible complete metric (the two most important properties of $\mathbb{R}$). See Section 1 of my DST notes for a gentle and loving introduction to Polish spaces.