Math 574: Set Theory HOMEWORK 7 **Due: Apr 5 and 6**

1. [1]Prove:

(a) $V_n = L_n$ for each $n \in \omega$; in particular, $V_\omega = L_\omega$.

(b) $V_{\omega+1} \neq L_{\omega+1}$, in fact, $|V_{\omega+1}| > |L_{\omega+1}|$.

REMARK: This is true even when $\mathbf{V} = \mathbf{L}$.

2. Let $F(x)$ be a Δ_0 class-function and $R(y, \vec{z})$ be a Δ_0 class-relation. In class we proved that the relation $R(F(x), \vec{z})$ is Σ_1 in general. However, prove that for the following class-functions $F(x)$, $R(F(x), \vec{z})$ is Δ_0.

(a) $F(x) := \bigcup x$ and $F(x) := \bigcap x$.

(b) $F(x) := \mathrm{dom}(x)$ if x is a function, and $\emptyset$, otherwise. Also, same with $\mathrm{dom}(x)$ replaced by $\mathrm{im}(x)$.

(c) (Optional) $F(x)$ is an arbitrary Δ_0 class-function such that the relation $z \in F(x)$ is also Δ_0 and for some $n \in \mathbb{N}$ (a genuine finite number, not an element of V), $\forall x\, F(x) \subseteq \mathrm{cl}_n(x)$, where

$$\mathrm{cl}_n(x) := \underbrace{\cup \cup \ldots \cup}_{n \text{ times}} x.$$

3. Let $F(x)$ be a Σ_1 class-function and let $\mathbf{M}$ be a transitive model of a large enough finite fragment of ZF. Suppose that for each x in $\mathbf{M}$ there is y in $\mathbf{M}$ such that $(F(x) = y)^{\mathbf{M}}$ holds. Prove:

(a) $F(x)$ is absolute for $\mathbf{M}$.

(b) If $\varphi(y, \vec{z})$ is an absolute formula for $\mathbf{M}$, then so is $\exists y \big((y = F(x)) \wedge \varphi(y, \vec{z})\big)$.

REMARK: If you think this is absolutely trivial, you are right.

4. Prove that the following class-functions satisfy the hypothesis of Question 3:

(a) $F(x, n) := x^n$ if $n \in \omega$, and $\emptyset$, otherwise.

HINT: $y = x^n$ if and only if there is a certificate $c : \omega \to x$ such that $c(0) = \emptyset$ and for each $k < n \ldots$

(b) $F(x) := x^{<\omega}$.

CAUTION: The class-function $F(x) := x^\omega$ is very nonabsolute.

Conclude that these class-functions are absolute for transitive models of a large enough finite fragment of ZF.

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[1]Thanks to Christian Schulz for suggesting this question.