Math 574: Set Theory HOMEWORK 6 **Due: Mar 29 and 30**

Conventions.

- From now on, we will use boldface letter to denote classes, e.g., $\mathbf{V}, \mathbf{M}, \mathbf{OD}$.
- Instead of the term $\mathbf{V}$-*formula*, we will use the term $*$-*formula* to get rid of the presence of $\mathbf{V}$ from notation.
- Unless mentioned otherwise, $(\mathbf{V}, \in)$ is a model of ZF.

1. Prove:

(a) $|V_\omega| = \omega$.

(b) For each $*$-formula f, $f \in V_\omega$.

2. Follow the steps below to prove the following

MetaTheorem. *There is a finite $T \subseteq$ ZF such that each transitive proper class $\mathbf{M} \subseteq \mathbf{V}$ satisfying T is a model of* ZF.

Let $\mathbf{M} \subseteq \mathbf{V}$ be a transitive proper class.

(a) If $\mathbf{M}$ satisfies a large enough finite fragment of ZF, then:

(i) $V_\alpha^{\mathbf{M}} = V_\alpha \cap \mathbf{M}$, for each ordinal α in $\mathbf{Ord}^{\mathbf{M}}$.

(ii) $\mathbf{Ord} \subseteq \mathbf{M}$;

HINT: Otherwise, $\mathbf{Ord}^{\mathbf{M}}$ is a set, and hence, so is $\mathbf{M}$ because $\mathbf{M} = \bigcup_{\alpha \in \mathbf{Ord}^{\mathbf{M}}} V_\alpha^{\mathbf{M}}$.

(iii) $V_\alpha \subseteq \mathbf{M}$ for all $\alpha \leqslant \omega$, in particular V_ω;

(iv) For each x in $\mathbf{V}$ if $x \subseteq \mathbf{M}$ then there is y in $\mathbf{M}$ such that $x \subseteq y$.

(b) Suppose that $\mathbf{M}$ satisfies the Comprehension Schema and has the property that for each x in $\mathbf{V}$, if $x \subseteq \mathbf{M}$ then there is y in $\mathbf{M}$ such that $x \subseteq y$. Then $\mathbf{M}$ is a model of ZF.

(c) Recall that the instance of Comprehension for a formula $\varphi(x, \vec{y})$ states the following[1]:

$$\forall \vec{y} \forall z \exists z' \left[x \in z' \leftrightarrow (x \in z \wedge \varphi(x, \vec{y})) \right].$$

Let $\psi(x, \alpha, f, \vec{p})$ be a formula which states that

(i) α is an ordinal,

(ii) f is a $*$-formula with $v_0 \in \text{var}(f)$,

(iii) $\vec{p} \in V_\alpha^n$, where $n := |\text{var}(f)| - 1$,

(iv) $x \in \text{Val}^*(f(v_0, \vec{p}), V_\alpha)$.

[1]Think of $\vec{y}$ as parameters.

Prove that if **M** satisfies the instance of Comprehension for ψ and a large enough finite fragment of ZF, then **M** satisfies the full Comprehension Schema, i.e. **all** instances of Comprehension.

Hint: Apply the Reflection principle (the general version from last homework) to $\mathbf{M} = \biguplus_\alpha V_\alpha^{\mathbf{M}}$ and the formula $\varphi(x, \vec{y})$ whose instance of Comprehension you want to prove in **M**. Use the Comprehension for ψ with $f := \ulcorner \varphi \urcorner$.

(d) Deduce the theorem.

3. Prove that any extension $T \supseteq \mathrm{ZF}$ (for example $T = \mathrm{ZF}$ or $T = \mathrm{ZFC}$) is not finitely axiomatizable[2]. Compare this to the metatheorem in Question 2, what is going on?

Hint: Let T_0 be a hypothetical finite axiomatization of T and let α be the least ordinal such that V_α is a model of T_0.

4. Prove that the following are equivalent:

(1) $\mathbf{V} = \mathbf{OD}$.

(2) $\mathbf{V} = \mathbf{HOD}$.

(3) **OD** is transitive.

(4) Extensionality holds in **OD**.

Hint: For (4)⇒(1), show that $V_\alpha \subseteq \mathbf{OD}$ for each α. This would follow from V_α in **OD** and $V_\alpha \cap \mathbf{OD}$ in **OD**.

[2] A theory T is said to be *finitely axiomatizable* if there is a finite theory T_0 such that for each sentence φ,

$$T \models \varphi \iff T_0 \models \varphi.$$