Math 574: Set Theory HOMEWORK 5 **Due: Mar 15 and 16**

1. Show that in $V_{\omega\cdot 2}$ it is not true that every well-ordering is isomorphic to a ordinal.

HINT: Consider the set $2\times\omega$. You need to use the fact that $\alpha \notin V_\alpha$, which is proven by an easy induction.

AFTERTHOUGHT: Which instance of Replacement fails in $V_{\omega\cdot 2}$?

2. (a) Prove that there is a first-order formula $\varphi(x)$ such that for any ordinal α, if $\varphi(x)$ is absolute for V_α then $\alpha = \aleph_\alpha$.

HINT: Include in $\varphi(x)$ the axioms from ZF that are needed to guarantee that the class-function $\beta \mapsto \aleph_\beta$ is defined for all ordinals. Next, include the instance of Replacement for this class function. Finally, include a condition on x such that $\varphi(x)$ being absolute for V_α ensures that the class-function $\beta \mapsto \aleph_\beta$ is absolute for V_α.

(b) Prove that there is a first-order formula $\psi(x,y)$ such that if $\psi(x,y)$ is absolute for any transitive nonempty set M, then $M = V_\lambda$ for some limit ordinal λ.

HINT: Similar to the previous part, but now you want the absoluteness of $\psi(x,y)$ for M to guarantee that the class-function $\beta \mapsto V_\beta$ is defined for all ordinals and is absolute.

3. Prove the following:

General Reflection Principle (ZF $^-$). *Let U be a model of* ZF$^-$ *and $\alpha \mapsto W_\alpha : \mathrm{Ord} \to U$ be a class function such that it is*

(i) *increasing:* $\alpha < \beta \implies W_\alpha \subseteq W_\beta$,

(ii) *continuous:* λ *limit* $\implies W_\lambda := \bigcup_{\alpha<\lambda} W_\beta$.

Let W denote the class $\bigcup_{\alpha \text{ in Ord}} W_\alpha$. For every first-order formula $\varphi(\vec{x})$ without parameters and every ordinal α, there is a limit ordinal $\beta > \alpha$ such that $\varphi(\vec{x})$ and all of its subformulas are absolute between W_β and W, i.e.

$$\forall \vec{a} \in W_\beta^{|\vec{x}|} \ \left(\varphi^{W_\beta}(\vec{a}) \leftrightarrow \varphi^W(\vec{a})\right)$$

4. Let $\varphi(\vec{x})$ be a first-order formula in the signature $(\in)$, all variables in $\vec{x} := (x_1, x_2, \ldots, x_n)$ are free.

(a) Give the full definition of the translation $\ulcorner\varphi\urcorner$ of φ to a U-formula. (We did one clause of this inductive definition in lecture.)

(b) Let A be a set. For any U-formula f with $k \in \omega$ variables, let

$$\mathrm{Val}^*(f,A) := \left\{\vec{a} \in A^k : \exists \delta \in \mathrm{Val}(f,A) \text{ and } \vec{a} = \delta\big(\mathrm{var}(f)\big)\right\}.$$

This is just the usual definition of the subset of A^k that a formula with k free variables defines, as opposed to defining a set of functions $\delta : \mathrm{var}(f) \to A$ like we did in class. Prove that

$$\mathrm{Val}^*(\ulcorner\varphi\urcorner, A) = \left\{\vec{a} \in A^n : \varphi^A(\vec{a})\right\}.$$

In other words, this says that the first-order formula $\varphi(\vec{x})$ carves out (i.e. defines) the same subset of A^n as the U-formula $\ulcorner\varphi(\vec{x})\urcorner$.

Note: When writing A^n above, we abused the notation and identified our $n \in \mathbb{N}$ with the corresponding ordinal (a point) in U.

5. Prove the following:

U-**Löwenheim–Skolem Theorem** (ZFC)**.** *For any sets $P \subseteq A$ there is a set $P \subseteq B \subseteq A$ with $|B| \leqslant |P| + \aleph_0$ such that for any U-formula $f(\vec{x})$ and any $\vec{b} \in B^{|\vec{x}|}$*

$$B \models f(\vec{b}) \iff A \models f(\vec{b}).$$

Remark: We have not defined everything involved in this question, but making sense of it is part of it.

Hint: The proof is given on page 82 of Rosendal's Note 3, but I suggest trying to prove it first without looking.

6. Listen to "Pyramid Song" by Radiohead and email me (Anush) a sentence describing what you thought of it. You don't have to present this in a problem session.