Complete, 9:57pm, Mar 4.

Math 574: Set Theory Homework 4 **Due: Mar 8 and 9**

1. Prove the following metatheorem: Every finite class of U is a set (i.e. is realized by an element of U). Explain why this is a metatheorem (i.e. not a first-order statement in U).

Remark: This question should have been assigned much earlier, but it didn't occur to me until now.

2. Prove directly that Pairing and Powerset axioms hold in V.

3. Prove that the following class-relations and class-functions, defined in $T := \mathrm{ZF}^- - \mathrm{Pow} - \mathrm{Infty}$, are all Δ_0 and hence absolute for any transitive model of T. You may choose to prove only half of these, but please choose the ones you doubt most.

Notation and terminology. Below, for class-functions, we just write $F(\vec{x})$ instead of $\vec{x} \mapsto F(\vec{x})$. Recall that a class-function is Δ_0 if, by definition, its graph $y = F(\vec{x})$ is Δ_0. This includes the 0-ary functions, i.e. constants such as $\emptyset, \omega$.

(a) $\{x\}$.

(b) (x, y).

(c) z is an ordered pair.

(d) $\emptyset$.

(e) $x \cup y, x \cap y, x \setminus y$.

(f) $\bigcup x$, $\bigcap x$.

(g) $S(x) := x \cup \{x\}$.

(h) x is a successor (of some set).

(i) x is transitive.

(j) $\in$ is a linear order on x.

(k) $x \times y$.

(l) R is a relation, i.e. is a set of ordered pairs.

(m) $\mathrm{dom}(R) := \{x : \exists y\ (x, y) \in R\}$ and $\mathrm{ran}(R) := \{y : \exists x\ (x, y) \in R\}$, i.e. the class-functions $F(R, X)$ (resp. $F(R, Y)$) defined by setting it to hold if R is a relation (i.e. a set of ordered pairs) and $X = \mathrm{dom}(R)$ (resp. $Y = \mathrm{ran}(R)$).

Remark: The definition of $\mathrm{dom}(R)$ is written as a class on purpose: you have to rewrite it so that it is a set and the class-function $R \mapsto \mathrm{dom}(R)$ is Δ_0.

(n) f is a function.

(o) $f(x)$, i.e. the class-function $F(f, x, y)$, which is set to hold exactly when f is a function, x is in $\mathrm{dom}(f)$, and $y = f(x)$.

(p) f is a one-to-one function.

4. Describe your own understanding of how it is possible that in a model $(U, \in)$ of ZFC, then the structure $(\omega^U, \in^U)$ may not be isomorphic to $(\mathbb{N}, <)$. Here, ω^U is the element of U that U defines as its first limit ordinal.

5. Consider a train which starts its journey from 0 and goes up to ω_1, making a stop at every ordinal on its way. At each station $\alpha < \omega_1$, one person gets off the train if it is not empty, and (regardless) α-many people get on the train. Prove that the train arrives empty at ω_1.

Hint: Fodor's lemma.

6. Prove that if κ is a (strongly) inaccessible cardinal, then the structure $(V_\kappa, \in)$ satisfies Extensionality, Set Existence, Pairing, Union, Powerset, Infinity, and Foundation axioms.