10:50pm, Feb 15.

Math 574: Set Theory HOMEWORK 3 **Due: Feb 22 and 23**

1. Let κ be an infinite cardinal and let $(A_\alpha)_{\alpha<\kappa}$ be a sequence of sets, each having cardinality at most κ. Carefully prove that $\left|\bigcup_{\alpha<\kappa} A_\alpha\right| \leqslant \kappa$, pinpointing *every* instance of AC you use.

2. Let I be a set and $(A_i)_{i\in I}$ be a sequence of sets. We denote by $\bigsqcup_{i\in I} A_i$ the disjoint union of A_i, i.e. $\bigsqcup_{i\in I} A_i := \{(i,a) : a \in A_i, i \in I\}$. Follow the steps below to prove König's theorem and its corollary proven in class.

Theorem (König). *Let I be a set and $(A_i)_{i\in I}, (B_i)_{i\in I}$ be sequences. If $|A_i| < |B_i|$ for all $i \in I$, then $\left|\bigsqcup_{i\in I} A_i\right| < \left|\prod_{i\in I} B_i\right|$.*

(a) Bofore proving the theorem, use it to conclude what we proved in class directly: $\kappa^{\mathrm{cof}(\kappa)} > \kappa$.

(b) Now use AC (more than once) to define an injection $\bigsqcup_{i\in I} A_i \hookrightarrow \prod_{i\in I} B_i$.

(c) Suppose towards a contradiction that there is a surjection $g : \bigsqcup_{i\in I} A_i \twoheadrightarrow \prod_{i\in I} B_i$ and define $x \in \prod_{i\in I} B_i$ that is not in the image of g by choosing the value $x(i)$ such that it precludes x from being in the g-image of $\{i\} \times A_i$. The main point is that for each $i \in I$, the map $g_i : A_i \to B_i$ defined by $a \mapsto g(i,a)(i)$ is not surjective, so one can choose a value from $B_i \setminus g_i[A_i]$ as $x(i)$.

3. Prove that AF is equivalent to the inexistence of an $\in$-decreasing ω-sequence[1], i.e. $(x_n)_{n<\omega}$ such that $x_{n+1} \in x_n$ for each $n < \omega$. Note: these x_n need not be pairwise distinct, e.g., it could be that $x_n = x$ for all $n < \omega$.

4. Prove that for each ordinal α, $\alpha \in V_{\alpha+1}$.

5. Prove that if a set x is not in V, then there is an $\in$-decreasing ω-sequence starting with x.

6. A set A is said to be *extensional* if for each $x, y \in A$, $x \cap A = y \cap A$ implies $x = y$. Provide an example and a counterexample.

7. The purpose of this question is to illustrate the counter-intuitiveness of AC.

Prisoners and hats. ω-many prisoners were sentenced to death, but they could get out under the following condition. On the day of the execution they will be lined up, i.e. enumerated $(p_n)_{n\in\mathbb{N}}$, so that each of them can see everyone with higher index (not themselves though). Each prisoner will have a red or blue hat put on him/her without being told which color it is. On command, all the prisoners (at once) make a guess as to what color they think their hat is. If all but finitely many guess correctly, they all go home free; otherwise all of them are executed. The good news is that the prisoners think of a plan the day before the execution, and indeed, all but finitely many guess correctly the next day, so everyone is saved. How do they do it?

HINT: None of the prisoners sees the whole binary sequence, but they all see it up to a certain notion of equivalence. Choose a representative from each equivalence class.

[1] For any set I, by an *I-sequence* we mean a element of the universe U that is a function on I.