Math 574: Set Theory HOMEWORK 2 **Due: Feb 15 and 16**

1. Recall that ordinals α and β are equal if they are isomorphic as orderings. Use this to prove the following facts of ordinal arithmetic, letting α, β, γ denote ordinals.

(a) *Associativity.* $\alpha + (\beta + \gamma) = (\alpha + \beta) + \gamma$ and $\alpha \cdot (\beta \cdot \gamma) = (\alpha \cdot \beta) \cdot \gamma$.

(b) *Identity elements.* $\alpha + 0 = \alpha = 0 + \alpha$ and $\alpha \cdot 1 = \alpha = 1 \cdot \alpha$.

(c) *Left-distributivity.* $\alpha \cdot (\beta + \gamma) = \alpha \cdot \beta + \alpha \cdot \gamma$.

(d) *Commutativity on ω.* $+$ and $\cdot$ are commutative on ω; in particular, right-distributivity holds as well.

(e) *Continuity.* If β is a limit ordinal, then $\alpha + \beta = \sup_{\gamma < \beta}(\alpha + \gamma)$ and $\alpha \cdot \beta = \sup_{\gamma < \beta}(\alpha \cdot \gamma)$.

2. Every ordinal is of the form $\lambda + n$, where λ is a limit ordinal or 0 and n is a natural number.

3. Let α be an ordinal and prove the following.

(a) α is infinite $\iff$ $1 + \alpha = \alpha$.

(b) $\alpha + 1$ (where $+$ is ordinal addition) is the successor of α; in particular, $\alpha + 1 > \alpha$.

(c) α is not a successor $\iff$ $2 \cdot \alpha = \alpha$.

(d) $\alpha \neq 0 \iff \alpha \cdot 2 > \alpha$.

(e) α is infinite $\iff$ $(1+1) \cdot \alpha < \alpha + \alpha$.

HINT: Use Question 2.

4. Letting α denote an ordinal and n a natural number, prove the following facts about ordinal exponentiation.

(a) $\exp(\alpha, n) = \underbrace{\alpha \cdot \alpha \cdot \ldots \cdot \alpha}_{n \text{ times}}$. In particular, $\exp(\alpha, n)$ is isomorphic to the reverse lexicographic ordering on the Cartesian product α^n.

(b) $\exp(2, \omega) = \omega$. Thus, $\exp(2, \omega + 1) = \omega + \omega$.

(c) For a set S, let $S^{<\mathbb{N}}$ denote the set of all finite tuples of elements of S, i.e. $S^{<\mathbb{N}} := \bigcup_{n \in \mathbb{N}} S^n$. Equip $\mathbb{N}^{<\mathbb{N}}$ with an ordering $\prec$ so that $\left(\mathbb{N}^{<\mathbb{N}}, \prec\right)$ is isomorphic to $\exp(\omega, \omega)$.

HINT: This is almost trivial, once you recall that $\exp(\omega, \omega) := \bigcup_{n < \omega} \exp(\omega, n)$.

5. For sets A, B, writing $A \hookrightarrow B$ means that there is an injection of A into B. Prove directly (without using the Cantor–Schröder–Bernstein theorem) that for a cardinal λ and an ordinal κ,

$$\lambda \hookrightarrow \kappa \iff \lambda \leqslant \kappa.$$

Hint: First realize that for any ordinals α, β, if $g : \alpha \hookrightarrow \beta$ is order-preserving, then $\alpha \leqslant \beta$. Now let $f : \lambda \hookrightarrow \kappa$ be an injection, which may not be order-preserving. We replace this with an order-preserving map $f' : \lambda' \hookrightarrow \kappa$, where λ' is some ordinal $\geqslant \lambda$. How do we get such f' and λ'? Look at $f[\lambda]$, it is well-ordered and hence is isomorphic to an ordinal λ'.

6. **Cantor's antidiagonalization.** Let X and $R \subseteq X^2$ be sets and for $x \in X$, denote by R_x the *fiber of R over x*, i.e.

$$R_x := \{y \in X : (x, y) \in R\}.$$

Let ∇_R denote the *antidiagonal* of R, i.e. $\nabla_R := \{x \in X : (x, x) \notin R\}$. Prove that ∇_R is not a fiber of R, i.e. there is no $x \in X$ with $\nabla_R = R_x$.

7. Give an AC-free proof of Hartog's theorem, which states that for every set X there is a cardinal κ such that $\kappa \not\hookrightarrow X$. Conclude the existence of an uncountable cardinal (in ZF, i.e. without AC).

Remark: Most mathematicians assume one has to use AC to get an uncountable cardinal.

Hint: Let A be the set of all ordinals that admit a bijection with a subset of X. Show that A itself is an ordinal and that $A \not\hookrightarrow X$.