4:30pm, Apr 24.

Math 574: Set Theory HOMEWORK 10 **Due: Apr 26 and 27**

Definition. Say that a poset $(\mathbb{P}, \leqslant)$

- is *separative*[1] if for any $p, q \in \mathbb{P}$, $p \nleqslant q \implies \exists r \leqslant p$ with $r \perp q$.
- has the *countable chain condition*[2] (c.c.c.) if every antichain is countable.

Below, let M be a countable transitive model of ZF and let $(\mathbb{P}, \leqslant) \in M$ be a poset.

1. Supposing that $(\mathbb{P}, \leqslant)$ is separative, show that no M-generic nonprincipal[3] ultrafilter $G \subseteq \mathbb{P}$ is an element of M.

HINT: Show that $\mathbb{P} \setminus G$ is dense.

2. For sets A, B with B having at least two elements, observe that the poset $\left([A \rightharpoonup B]^{<\omega}, \supseteq\right)$ is separative and, assuming that B is countable, prove that it has the c.c.c.

HINT: For c.c.c., apply the Δ-system lemma to the domains of the functions in a hypothetical uncountable antichain.

Observation (Use freely below)**.** *For any $a, b \in M$ and $p \in \mathbb{P}$,*

(i) $p \Vdash \neg \underline{a} \in \underline{b}$ *if and only if* $p \Vdash \underline{a} \notin \underline{b}$;

(ii) $p \Vdash \neg \underline{a} \neq \underline{b}$ *if and only if* $p \Vdash \underline{a} = \underline{b}$.

3. Let $G \subseteq \mathbb{P}$ an M-generic ultrafilter. Prove that for any $a, b \in M$,

$$a[G] \neq b[G] \iff \exists p \in G \text{ with } p \Vdash \underline{a} \neq \underline{b}.$$

In your proof by induction on the pair-rank ordering $\prec$, you may assume that for any pair $(a', b') \prec (a, b)$,

$$a'[G] \in b'[G] \iff \exists p \in G \text{ with } p \Vdash \underline{a'} \in \underline{b'}.$$

Definition. In a topological space X, a set $A \subseteq X$ is said to be

- *somewhere*[4] *dense* if $A \cap U$ is dense in some nonempty open $U \subseteq X$;
- *nowhere dense* if it is not somewhere dense;
- *meager* if it is a countable union of nowhere dense sets;
- *comeager* if its complement is meager;
- *Baire measurable* if it "meager away from being open", i.e. $A \triangle U$ is meager for some open (possibly empty) $U \subseteq X$;
- G_δ (resp. F_σ) if it is a countable intersection (resp. union) of open (resp. closed) sets.

Notation. Let X be a topological space and $A \subseteq X$.

- Denote the *boundary* A by ∂A, i.e. $\partial A := \overline{A} \setminus \mathrm{Int}(A)$.

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[1] Recall that this condition was needed in 1(d) of HW9.

[2] I would name it *countable antichain condition*, of course.

[3] Call an ultrafilter $G \subseteq \mathbb{P}$ *principal* if $G = \delta_p := \{q \in \mathbb{P} : q \geqslant p\}$ for some $p \in \mathbb{P}$.

[4] In general, in a topological space, "somewhere" stands for "in some nonempty open set".

- For an open set U, write $U \Vdash A$ and say that U *forces* A, if $A \cap U$ is comeager in U, i.e. $U \setminus A$ is meager.

4. Let X be a topological space.

(a) Prove: A set $A \subseteq X$ is nowhere dense $\iff$ $\overline{A}$ is nowhere dense $\iff$ A^c contains an dense open set. In particular, a set is comeager if and only if it contains a countable intersection of dense open sets.

(b) Prove: For a closed set $K \subseteq X$, ∂K is nowhere dense. Hence, for an open set U, ∂U is nowhere dense.

(c) Conclude: Baire measurable sets are closed under complements.

(d) Prove: Baire measurable sets are closed under countable unions.

(e) Conclude: Baire measurable sets form a σ-algebra and hence Borel sets are Baire measurable.

Definition. A topological space is said to be *Baire* if every nonempty open set is nonmeager.

5. Prove that a topological space is Baire if and only if every comeager set is dense if and only if every countable intersection of dense open set is dense.

Baire Catogory Theorem[5]**.** *Completely metrizable topological spaces are Baire. In particular, Polish spaces are Baire.*

6. Let X be a topological space and let $\mathcal{V}$ be a weak basis[6] for X. Prove:

(a) **Baire alternative:** For any Baire measurable $A \subseteq X$, either A is meager or there is a nonempty open $U \subseteq X$ with $U \Vdash A$.

(b) For sets $A_n \subseteq X$, $n \in \mathbb{N}$, and open $U \subseteq X$,

$$U \Vdash \bigcap_n A_n \iff \forall n \in \mathbb{N}\, (U \Vdash A_n).$$

(c) If X is Baire, $A \subseteq X$ is Baire measurable, and $U \subseteq X$ is nonempty open, then

$$U \Vdash A^c \iff \forall V \subseteq U\, (V \nVdash A),$$

where V varies over $\mathcal{V}$.

(d) (Optional) If X is Baire, the sets $A_n \subseteq X$, $n \in \mathbb{N}$, are Baire measurable, and U is nonempty open, then

$$U \Vdash \bigcup_n A_n \iff \forall V \subseteq U\, \exists W \subseteq V\, \exists n \in \mathbb{N}\, (W \Vdash A_n).$$

where V, W vary over $\mathcal{V}$.

Hint: Unions are expressed via intersections and complements.

[5]For a proof of this, see, for example, Theorem 6.12 in my DST notes.

[6]Recall that a *weak basis* for a topological space X is a collection $\mathcal{V}$ of nonempty open sets such that every nonempty open set $U \subseteq X$ contains at least one $V \in \mathcal{V}$.