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Math 574: Set Theory HOMEWORK 1 **Due: Feb 1 and 2**

1. (a) Write down an explicit first-order formula $\varphi(x)$ that holds in a model $(U, \in)$ of set theory if and only if x is an ordered pair, i.e. is of the form (x_0, x_1).

(b) Prove that for all x_0, x_1, y_0, y_1, $(x_0, x_1) = (y_0, y_1)$ if and only if $x_0 = y_0$ and $x_1 = y_1$.

(c) Let n be a fixed natural number (e.g. $n = 7$). For all $x_1, x_2, \dots, x_n$, define/show existence of the set $\{x_1, x_2, \dots, x_n\}$.

2. (a) For any sets A, B, define and prove the existence of the sets $A \setminus B$, $A \cap B$, and $\bigcap A$.

(b) For any sets I, X and a function $a : I \to X$, prove the existence of the sets $\bigcup_{i \in I} a(i)$, $\bigcap_{i \in I} a(i)$, and $\prod_{i \in I} a(i)$.

3. Explain very *precisely* (without hand-waving, tap dancing, or the like) what classes are and how they are different from sets.

4. Prove that a class function whose domain is a set is actually a (set) function.

5. For classes C, D, if C is proper and there is a class injection F of C into D, then D is proper. In fact, show that the assumption of the injectivity can be relaxed to requiring that F is a set-to-one class function, i.e. for every y (in U), $F^{-1}(y) := \{x : F(x) = y\}$ is a set.

6. (a) Prove that any nonempty subclass C of Ord has an $\in$-least element.

(b) Deduce that a transitive set X of ordinals is an ordinal. You may not use the statement that $\bigcup X$ is an ordinal because the proof of the latter uses the former.

7. State and prove the class version of the theorem that every well-ordering is isomorphic to an ordinal. This would be a *theorem-schema*, i.e. one theorem for each class in the hypothesis.

8. State and prove the class version of the definition by induction theorem. Again, this would be a theorem-schema.

REMARK: This can be used to show to build a class injection of Ord into a given class C, thus showing that C is proper (c.f. Question 5).