Math 540: Real Analysis HOMEWORK 9 **Due date: Apr 11 (Tue)**

0. (Reflection) Write a short (no more than 3/4 of a page) essay discussing the following:
- the definition of abstract integral and its difference from Riemann integral on $\mathbb{R}$;
- various modes of convergence and their relationship;
- the magic of Fubini's theorem.

Exercises from Bass's textbook. 15.12

Hint (for 15.12). g vanishes at ∞ in L^q-norm, so we can focus on a set of finite measure and apply Egorov's theorem.

Definition. Let X be a set and $\mathcal{C} \subseteq \mathscr{P}(X)$. Call a function $\varphi : X \to \bar{R}$ (*rational*) $\mathcal{C}$*-simple* if it is a finite (rational) linear combination of characteristic functions of sets from $\mathcal{C}$.

Definition. Call a σ-algebra $\mathcal{M}$ *countably generated* if $\mathcal{M} = \sigma(\mathcal{C})$ for some countable $\mathcal{C} \subseteq \mathcal{M}$.

1. Prove that, for any measure space and $p \in [1, \infty)$, simple functions are dense in L^p, i.e. for every $f \in L^p$ and $\varepsilon > 0$, there is a simple function φ with $\|f - \varphi\|_{L^p} < \varepsilon$.

2. Let $p \in [1, \infty)$ and let $(X, \mathcal{M}, \mu)$ be a σ-finite measure space, where the σ-algebra $\mathcal{M}$ is generated by a collection $\mathcal{C} \subseteq \mathcal{M}$.

(a) Let $\mathcal{A}$ be the algebra generated by $\mathcal{C}$ and prove that rational $\mathcal{A}$-simple functions are dense in L^p.

(b) Conclude that if $\mathcal{M}$ is countably generated, then L^p is separable. In particular, $L^p(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d), \lambda)$ and $\ell^p(\mathbb{N})$ are separable.

(c) Prove that if there are infinitely many disjoint sets in $\mathcal{M}$ of positive measure, then L^∞ is not separable. In particular, $L^\infty(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d), \lambda)$ and $\ell^\infty(\mathbb{N})$ are not separable.

3. Let $(X, \mathcal{M}, \mu)$ be a measure space and $0 < p < q < \infty$. Prove the following.

(a) If $\mathcal{M}$ contains sets of arbitrarily small positive measure, then $L^p \nsubseteq L^q$.

(b) If $\mathcal{M}$ contains sets of arbitrarily large finite measure, then $L^p \nsupseteq L^q$.

Hint. For either part, take disjoint sets $(A_n)_n$ of positive finite measure (with an additional smallness or largeness condition) and let $f|_{A_n}$ be an appropriately chosen constant.

Remark. The converses of both parts are also true.

4. (About L^∞) Let $(X, \mathcal{M}, \mu)$ be a measure space and prove the following.

(a) Hölder's inequality for $(1, \infty)$: for any measurable functions f, g,

$$\|fg\|_1 \leq \|f\|_1 \|g\|_\infty.$$

If $\|f\|_1, \|g\|_\infty < \infty$, then $\|fg\|_1 = \|f\|_1 \|g\|_\infty$ if and only if $|g(x)| = \|g\|_\infty$ for a.e. $x \in \operatorname{supp}(f) := \{x \in X : f(x) \neq 0\}$.

(b) $\|\cdot\|_\infty$ is a norm.

(c) $f_n \to_{L^\infty} f$ (i.e. $\|f_n - f\| \to 0$) if and only if $f_n \to f$ uniformly on a conull set (i.e. there is a conull set $Y \subseteq X$ such that $f_n|_Y \to_u f|_Y$).

(d) L^∞ is a Banach space.

(e) Simple functions are dense in L^∞ (in the $\|\cdot\|_\infty$-norm, of course).

(f) For any $p < \infty$ and $f \in L^p \cap L^\infty$, $\|f\|_\infty = \lim_{q \to \infty} \|f\|_q$.