Math 540: Real Analysis HOMEWORK 8 **Due date: Mar 21 (Tue)**

Exercises from Bass's textbook. 11.8, 11.9, 11.10, 11.15, 11.16

Hint (for 11.8). Measurable sets are approximated by boxes. f is absolutely continuous in L^1-norm (Problem 2(b) in Homework 7).

Hint (for 11.9). Use Tonelli's theorem to prove the integrability of the function.

Hint (for 11.16). Treat the sum as integral over $\mathbb{N}$ with the counting measure. It is enough to solve the problem with $\mathbb{R}$ replaced by any interval (a, b).

1. Show that the function $f : [0,1]^d \to [0,\infty]$ defined by $f(x) = ||x||_1^{-p}$ is integrable for any real $p < d$, where $x := (x_1, x_2, \ldots, x_d)$ and $||x||_1 := \sum_{i=1}^d |x_i|$.

2. Let $(X, \mathcal{M}, \mu)$ be a σ-finite measure space, $f : X \to [0,\infty]$ a non-negative measurable function, and let λ denote the Lebesgue measure on $\mathbb{R}$.

(a) Show that the set $G_f := \{(x, y) \in X \times [0,\infty] : y \le f(x)\}$ is $\mathcal{M} \otimes \mathcal{B}(\mathbb{R})$-measurable and prove that *the integral of f is equal to the area under its graph*, namely:

$$(\mu \times \lambda)(G_f) = \int f \, d\mu$$

Hint. For the measurability of G_f, use Problem 3 of Homework 4.

(b) Derive the so-called *Namioka trick*:

$$\int f \, d\mu = \int_0^\infty \mu(\{x \in X : y \le f(x)\}) \, d\lambda(y).$$