Math 540: Real Analysis HOMEWORK 7 **Due date: Mar 14 (Tue)**

Exercises from Bass's textbook. 7.16

Below let $(X, \mathcal{M}, \mu)$ be a measure space, let f and $(f_n)_n$ be measurable functions.

Definition. Say that f is *absolutely continuous in L^1-norm* if for every $\varepsilon > 0$ there is $\delta > 0$ such that for every measurable $A \subseteq X$, $\mu(A) < \delta \Rightarrow \int_A |f| d\mu < \varepsilon$. Say that f *vanishes at ∞ in L^1-norm* if for every $\varepsilon > 0$ there is a measurable $A \subseteq X$ with $\mu(A) < \infty$ such that $\int_{A^c} |f| d\mu < \varepsilon$.

Definition. Let $\mathcal{F}$ be a collections (or a sequence) of measurable functions. Call $\mathcal{F}$ *uniformly absolutely continuous* if for every $\varepsilon > 0$ there is $\delta > 0$ such that for every measurable $A \subseteq X$, $\mu(A) < \delta \implies \forall f \in \mathcal{F} \int_A |f| d\mu < \varepsilon$. Say that $\mathcal{F}$ *uniformly vanishes at ∞ in L^1-norm* if for every $\varepsilon > 0$ there is a measurable $A \subseteq X$ with $\mu(A) < \infty$ such that $\forall f \in \mathcal{F} \int_{A^c} |f| d\mu < \varepsilon$.

1. Let f and $(f_n)_n$ be measurable. Prove that if $f_n \to f$ approximately uniformly, then $f_n \to f$ a.e. and in measure.

2. Let $f \in L^1$ and prove the following.

(a) f vanishes at ∞ in L^1-norm.

Hint. Look at the sets $\left\{x \in X : \frac{1}{n} \leqslant |f(x)| < n\right\}$.

(b) f is absolutely continuous in L^1-norm.

Hint. Arguing by contradiction, get sets A_n with $\mu(A_n) < 2^{-n}$ and $\int_{A_n} |f| \geqslant \varepsilon$, and apply DCT.

3. Let $(f_n)_n \subseteq L^1$ and prove that $(f_n)_n$ converges in L^1-norm if and only if $(f_n)_n$ converges in measure, is uniformly absolutely continuous in L^1-norm, and uniformly vanishes at ∞ in L^1-norm.

Hint. For $\Leftarrow$ direction, given $\varepsilon > 0$, get a set A from uniformly vanishing at ∞ in L^1-norm property and get $\delta > 0$ from uniform absolute continuity, and then apply convergence in measure with appropriately chosen $\alpha > 0$ and this δ.

4. Give examples of sequences $(f_n)_n$ and $(g_n)_n$ of L^1-functions that converge both a.e. and in measure, but $(f_n)_n$ is not uniformly absolutely continuous in L^1-norm and $(g_n)_n$ does not vanish at ∞ in L^1-norm. In particular, neither of these sequences converge in L^1-norm.

5. Prove that every non-null subset $A \subseteq \mathbb{R}$ contains a Lebesgue non-measurable set.

Hint. Argue that it's enough to consider a bounded A, after which the exact same argument as for $A = (0, 1)$ works.

6.* (Optional) Let μ be the fair coin-flip measure on Cantor space $2^{\mathbb{N}}$. Show that there is a μ-non-measurable set $A \subseteq 2^{\mathbb{N}}$.

Hint. Consider the equivalence relation E_0 on $2^{\mathbb{N}}$ of *eventual agreement of binary sequences*, i.e. for $x, y \in 2^{\mathbb{N}}$, $x E_0 y :\Leftrightarrow \forall^\infty n \; x(n) = y(n)$.

Notation. Let X, Y be sets and fix $x_0 \in X, y_0 \in Y$. For a set $C \subseteq X \times Y$, define its *fibers* C_{x_0} and C^{y_0} at x_0 and y_0, respectively, as follows:

$$C_{x_0} := \{y \in Y : (x_0, y) \in C\} \text{ and } C^{y_0} := \{x \in X : (x, y_0) \in C\}.$$

Similarly, for a function $f : X \times Y \to Z$, define $f_{x_0} : Y \to Z$ and $f^{y_0} : X \to Z$ by setting

$$f_{x_0}(y) := f(x_0, y) \text{ and } f^{y_0}(x) := f(x, y_0)$$

for $x \in X$ and $y \in Y$.

7. Let $(X, \mathcal{M})$ and $(Y, \mathcal{N})$ be measurable spaces and fix $x_0 \in X, y_0 \in Y$.

(a) Prove that for any $C \subseteq \mathcal{M} \otimes \mathcal{N}$, $C_{x_0} \in \mathcal{N}$ and $C^{y_0} \in \mathcal{M}$.

Hint. As usual, consider the collection $\mathcal{C}$ of sets C that have the desired property and show that $\mathcal{C} = \mathcal{M} \otimes \mathcal{N}$.

(b) Let $(Z, \mathcal{L})$ be a measurable space and prove that for any $(\mathcal{M} \otimes \mathcal{N}, \mathcal{L})$-measurable function $f : X \times Y \to Z$, $f_{x_0} : Y \to Z$ is $(\mathcal{N}, \mathcal{L})$-measurable and $f^{y_0} : X \to Z$ is $(\mathcal{M}, \mathcal{L})$-measurable.