Complete

Math 540: Real Analysis HOMEWORK 6 **Due date: Mar 7 (Tue)**

Exercises from Bass's textbook. 7.3, 7.5, 7.13

Hint (for 7.13). First compute the same limit but with $\int_\delta^1$ instead of $\int_0^1$ for any $\delta > 0$.

Notation. Fix a measure space $(X, \mathcal{M}, \mu)$ for all problems below. We write L^1 below to mean $L^1(X, \mathcal{M}, \mu)$ and, for a function $f \in L^1$, we put $\|f\|_1 := \int |f|$. For a sequence $(f_n)_n$ and a function f in L^1, we write $f_n \to_{L^1} f$ to mean that $(f_n)_n$ *converges to f in the L^1-norm*, i.e. $\|f - f_n\|_1 \to 0$.

1. Prove that for any integrable function $f : X \to \overline{\mathbb{R}} := [-\infty, +\infty]$, the set $\{x \in X : f(x) = \pm\infty\}$ is null and the set $\{x \in X : f(x) \neq 0\}$ is σ-finite.

2. Let $f \in L^1$, $(f_n)_n \subseteq L^1$. Suppose that $f_n \to f$ a.e. and $\int |f_n| \to \int |f|$.

(a) Prove that $f_n \to_{L^1} f$.

Hint. Use the generalized Dominated Convergence Theorem (Exercise 7.5 of Bass).

(b) Conclude that for any measurable $A \subseteq X$, $\int_A f_n \to \int_A f$.

3. Let $(f_n)_n$ be a sequence of nonnegative Lebesgue integrable functions on $\mathbb{R}$. Prove or give a counterexample to the following statements.

(a) $\displaystyle\int \limsup_{n\to\infty} f_n \geq \limsup_{n\to\infty} \int f_n$.

(b) If $f_n \to 0$ both pointwise and in the L^1-norm, then there is $g \in L^1$ such that $f_n \leq g$ for all $n \in \mathbb{N}$.