Complete

Math 540: Real Analysis HOMEWORK 5 **Due date: Feb 28 (Tue)**

Terminology. Let $(X, \mathcal{M}, \mu)$ be a measure space and $f : X \to \mathbb{R}$. Below, whenever we use the term continuous for f, we assume that X is also a metric space.

(i) For a set $A \subseteq X$, let $f\downharpoonright_A$ denote the restriction of f to domain A, and we say that $f\downharpoonright_A$ *is $\mathcal{M}$-measurable* if, for every Borel $B \subseteq \mathbb{R}$, $f^{-1}(B) \cap A \in \mathcal{M}$.

(ii) For a set $A \subseteq X$, say that $f\downharpoonright_A$ *is continuous* if it is continuous as a function from the metric space A to $\mathbb{R}$.

(iii) For $\varepsilon \geq 0$, call a set $A \subseteq X$ *co-ε* or *co-μ-ε* if A^c is contained in some set $B \in \mathcal{M}$ with $\mu(B) < \varepsilon$.

(iv) Call f *approximately $\mathcal{M}$-measurable* (resp. *approximately continuous*) if, for every $\varepsilon > 0$, $f\downharpoonright_A$ is $\mathcal{M}$-measurable (resp. continuous) for some co-ε set $A \in \mathcal{M}$.

Notation. For a measure space $(X, \mathcal{M}, \mu)$, let $\overline{\mathcal{M}}$ denote the completion of $\mathcal{M}$ with respect to μ. For metric spaces $(X, d_X), (Y, d_Y)$ and a function $f : X \to Y$, denote by $\mathrm{DC}(f)$ the set of discontinuity points of f, i.e.

$$\mathrm{DC}(f) := \{x \in X : \exists \varepsilon > 0 \ \forall \delta > 0 \ \exists y \in X \text{ with } d_X(x, y) < \delta \text{ such that } d_Y(f(x), f(y)) \geq \varepsilon\}.$$

1. Let $(X, \mathcal{M}, \mu)$ be a measure space and $f : X \to \mathbb{R}$ a function. Prove that the following are equivalent:

(1) $f\downharpoonright_A$ is $\mathcal{M}$-measurable for some conull set $A \subseteq X$.

(2) f is $\overline{\mathcal{M}}$-measurable.

(3) f is approximately $\mathcal{M}$-measurable.

2. Prove that a function $f : \mathbb{R}^d \to \mathbb{R}$ is Lebesgue-measurable if and only if it is approximately continuous (with respect to the Lebesgue measure).

3. Prove that a bounded function $f : [0, 1] \to \mathbb{R}$ is Riemann-integrable if and only if it $\mathrm{DC}(f)$ is null (with respect to the Lebesgue measure).

4. (Degrees of continuity) Below, everything is with respect to the Lebesgue measure.

(a) Give an example of a function $f : \mathbb{R} \to \mathbb{R}$ such that $f\downharpoonright_A$ is continuous for some conull $A \subseteq X$, yet $\mathrm{DC}(f) = \mathbb{R}$.

(b) For any Cantor set $C \subseteq [0, 1]$ of positive measure, show that $f := \mathbf{1}_C$ is approximately continuous, yet $f\downharpoonright_A$ is discontinuous for every conull set $A \subset \mathbb{R}$.

5. This is an alternative approach for proving the linearity of integral on simple functions. Let φ be a simple function on a measure space $(X, \mathcal{M}, \mu)$ and let

$$\varphi = \sum_{i<n} a_i \mathbf{1}_{A_i}$$

be an arbitrary representation of φ (the sets A_i may not be pairwise disjoint). Prove that

$$\int \varphi d\mu = \sum_{i<n} a_i \mu(A_i)$$

and conclude that the integral is a linear functional[1] on the space of simple functions.

[1]This means $\int (c\varphi + \psi) d\mu = c \int \varphi d\mu + \int \psi d\mu$, for $c \in \mathbb{R}$ and φ, ψ simple functions on X.