Math 540: Real Analysis HOMEWORK 4 **Due date: Feb 21 (Tue)**

1. Let λ denote the Lebesgue measure on $\mathbb{R}^d$ and let $A \subseteq \mathbb{R}^d$ denote a Lebesgue measurable set.

(a) Prove that λ is *translation invariant*, i.e. $A \subseteq \mathbb{R}^d$, $\lambda(A) = \lambda(x + A)$ for any $x \in \mathbb{R}^d$.

(b) Prove that it respects scaling, i.e. $\lambda(rA) = r^d \lambda(A)$ for any real $r > 0$.

(c) (Optional) For a $d \times d$ matrix M, let $P(M)$ denote the d-dimensional parallelepiped spanned by the columns of M. Prove that if $\tilde{I}$ is a column-addition elementary matrix[1], then $\lambda(P(\tilde{I})) = \lambda(P(I)) = 1$.

(d) More generally, let M be a $d \times d$ real-valued matrix and show that $\lambda(MA) = \det(M)\lambda(A)$, where $MA \coloneqq \{Mx : x \in A\}$.

Hint. Any matrix is a product of elementary matrices.

2. The purpose of this problem is to demonstrate that natural sets that come up in analysis are very often Borel.

Let (X, d_X) and (Y, d_Y) be metric spaces and let $f : X \to Y$ be an arbitrary function. A point $x \in X$ is called a *continuity point* of f if f is continuous at x.

(a) For a set $C \subseteq Y$, put $\operatorname{diam}(C) \coloneqq \sup_{x,y \in C} d_Y(x, y)$, and, for $x \in X$, put

$$\operatorname{osc}_f(x) \coloneqq \inf \{\operatorname{diam}(f(B)) : B \subseteq X \text{ an open ball containing } x\}.$$

Prove that for every $x \in X$, x is a continuity point of f if and only if $\operatorname{osc}_f(x) = 0$.

(b) Deduce that the continuity points of f form a G_δ subset of X.

3. Let $(X, \mathcal{M})$ be a measurable space. Below, you may use the result stated in class $\mathcal{B}(\mathbb{R}^2) = \mathcal{B}(\mathbb{R}) \otimes \mathcal{B}(\mathbb{R})$, even though we didn't quite finish the proof.

(a) $(Y_i, \mathcal{N}_i)_{i \in I}$ a sequence of measurable spaces, let $\pi_i : \prod_{i \in I} Y_i \to Y_i$ denote the projection function onto the i^{th}-coordinate for each $i \in I$ and put $\mathcal{N} \coloneqq \bigotimes_{i \in I} \mathcal{N}_i$. Prove that a function $f : X \to \prod_{i \in I} Y_i$ is $(\mathcal{M}, \mathcal{N})$-measurable if and only if, for each $i \in I$, the function $\pi_i \circ f : X \to Y_i$ is $(\mathcal{M}, \mathcal{N}_i)$-measurable.

(b) Conclude that a function $f : X \to \mathbb{C}$ is $\mathcal{M}$-measurable if and only if both $\operatorname{Re}(f)$ and $\operatorname{Im}(f)$ are $\mathcal{M}$-measurable.

(c) For each $i = 0, 1$, let $(Y_i, \mathcal{N}_i)$ be a measurable space and $f_i : X \to Y_i$ an $(\mathcal{M}, \mathcal{N}_i)$-measurable function. Let $(Z, \mathcal{S})$ be a measurable space and let $g : Y_0 \times Y_1 \to Z$ be an $(\mathcal{N}_0 \otimes \mathcal{N}_1, \mathcal{S})$-measurable function. Prove that the composition $g(f_0, f_1) : X \to Z$ is $(\mathcal{M}, \mathcal{S})$-measurable.

(d) Conclude that for any $\mathcal{M}$-measurable $f, g : X \to \mathbb{R}$, the functions $f + g$ and $f \cdot g$ are also $\mathcal{M}$-measurable.

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[1] *Column-addition matrices* are those obtained from the identity matrix by adding a scalar multiple of one column to another.