Math 540: Real Analysis HOMEWORK 3 **Due date: Feb 7 (Tue)**

0. (Reflection) Write a short (no more than 3/4 of a page) essay discussing the following:

- your understanding of the problem of constructing a measure that extends a given function on a collection of sets (e.g. the volume-function on boxes in $\mathbb{R}^3$); give an example of a function that is not extendable;
- how you would try constructing an extension if you were asked a week ago;
- how we did it: list the main steps of our construction, providing a one-sentence motivation/explanation for each.

Notation. Let $P(n)$ be a property of a natural number $n \in \mathbb{N}$. Define

$$\exists^\infty n P(n) :\Leftrightarrow \text{for arbitrarily large } n\ P(n) \text{ holds} :\Leftrightarrow \forall m\, \exists n \geqslant m\, P(n)$$
$$\forall^\infty n P(n) :\Leftrightarrow \text{for all large enough } n\ P(n) \text{ holds} :\Leftrightarrow \exists m \forall\, n \geqslant m\, P(n).$$

1. Let μ^* be a finite outer measure on X (i.e. $\mu^*(X) < \infty$) and let d_{μ^*} be the pseudo-metric on $\mathscr{P}(X)$ as defined in class, i.e. $d_{\mu^*}(A, B) := \mu^*(A \triangle B)$. Prove that the union function $\cup : \mathscr{P}(X) \times \mathscr{P}(X) \to \mathscr{P}(X)$ given by $(A, B) \mapsto A \cup B$ is jointly continuous, i.e. for any sequences $(A_n)_n$ and $(B_n)_n$ converging to A and B, respectively, the sequence $(A_n \cup B_n)_n$ converges to $A \cup B$.

2. Let $(X_n)_{n\in\mathbb{N}}$ be a partition of a set X and let $\mathcal{M}_n$ be a σ-algebra on X_n, for each $n \in \mathbb{N}$.

(a) Letting $\mathcal{M}$ denote the σ-algebra on X generated by $\bigcup_{n\in\mathbb{N}} \mathcal{M}_n$, prove that $\mathcal{M}$ is exactly the collection of all subsets of X of the form $\bigsqcup_{n\in\mathbb{N}} A_n$, where $A_n \in \mathcal{M}_n$ for each $n \in \mathbb{N}$.

(b) Now let μ_n be a measure on $\mathcal{M}_n$, for each $n \in \mathbb{N}$. Define a measure μ on $\mathcal{M}$ that extends every μ_n (i.e. $\mu|_{\mathcal{M}_n} = \mu_n$) and prove your answer. Also show that such a measure is unique.

3. (Correct proof of countable additivity of the extension) Let ρ be a finite premeasure on an algebra $\mathcal{A}$ (on a set X) and let $\mathcal{M}$ be the closure of $\mathcal{A}$ in the pseudo-metric $d := d_{\mu_\rho^*}$. The proof I gave in class of countable additivity of μ_ρ^* on $\mathcal{M}$ is incorrect: the errors coming from disjointification of the sets A_n add up to ∞. Fill in the details in the following correct proof.

Proof. By continuity, μ_ρ^* is finitely additive on $\mathcal{M}$ because it is finitely additive on $\mathcal{A}$ and $\mathcal{A}$ is dense in $\mathcal{M}$. But μ_ρ^* is also countably subadditive, which amplifies finite additivity to countable additivity. $\square$

4. (Borel-Cantelli Lemma) Let $(X, \mathcal{M}, \mu)$ be a measure space and let $\{B_n\}_{n\in\mathbb{N}} \subseteq \mathcal{M}$ be such that $\sum_{n\in\mathbb{N}} \mu(B_n) < \infty$. Prove that the set

$$B := \limsup_{n\in\mathbb{N}} B_n := \{x \in X : \exists^\infty n\, x \in B_n\}.$$

Prove that $B \in \mathcal{M}$ and $\mu(B) = 0$.

Remark. This lemma, although easy, is extremely useful.[1] Say you have bad sets B_n that you'd like each point $x \in X$ to eventually avoid (i.e. $\forall^\infty n\,(x \notin B_n)$). If your bad sets have summable measure (an indication of smallness), then good news: only a μ-null set of points in X are so unfortunate as to hit infinitely many of these bad sets.

5. Let ρ be a finite premeasure on an algebra $\mathcal{A}$ on a set X and let $\mathcal{M}$ be the closure of $\mathcal{A}$ in the pseudo-metric $d \coloneqq d_{\mu_\rho^*}$. Prove that for every $B \in \mathcal{M}$, there is $A \in \sigma(\mathcal{A})$ with $d(A, B) = 0$. Conclude that $(X, \mathcal{M}, \mu)$ is the completion of $(X, \sigma(\mathcal{A}), \mu)$.

Hint. Borel-Cantelli.

6. Let $(X, \mathcal{M}, \mu)$ be a σ-finite measure space and let $\mathcal{C} \subseteq \mathcal{M}$ be a collection that *generates* $\mathcal{M}$ (as a σ-algebra), i.e. $\sigma(\mathcal{C}) = \mathcal{M}$. Show that the algebra $\mathcal{A}$ generated by $\mathcal{C}$ is dense in $\mathcal{M}$ with respect to the pseudo-metric d_μ on $\mathcal{M}$ defined by $d_\mu(A, B) = \mu(A \triangle B)$. Conclude that if $\mathcal{M}$ is *countably generated* (i.e. admits a countable generating $\mathcal{C} \subseteq \mathcal{M}$), then the pseudo-metric space $(\mathcal{M}, d_\mu)$ is separable[2].

[1] I often call it the "fundamental theorem of measure theory".

[2] A metric/topological space is called *separable* if it admits a countable dense subset.