Math 540: Real Analysis HOMEWORK 2 **Due date: Jan 31 (Tue)**

1. Prove that there doesn't exist a probability measure μ on $\mathcal{B}(\mathbb{R})$ that is invariant under the translation action of $\mathbb{Z}$ on $\mathbb{R}$, i.e. for every $A \in \mathcal{B}(\mathbb{R})$ and $\gamma \in \mathbb{Z}$, $\mu(\gamma + A) = \mu(A)$.

For future reference, here are the definitions of *group action* and *invariant measure*.

Definition (Group action)**.** Let $(\Gamma, \cdot_\Gamma, 1_\Gamma)$ be a group and X a set. An action $\alpha : \Gamma \curvearrowright X$ is a map $\alpha : \Gamma \times X \to X$ satisfying the following properties, where we use the notation $\gamma \cdot_\alpha x$ instead of $\alpha(\gamma, x)$, for $\gamma \in \Gamma, x \in X$:

(i) Identity: $1_\Gamma \cdot_\alpha x = x$ for all $x \in X$.

(ii) Associativity: $\gamma_1 \cdot_\Gamma (\gamma_2 \cdot_\alpha x) = (\gamma_1 \cdot_\Gamma \gamma_2) \cdot_\alpha x$, for all $x \in X$ and $\gamma_1, \gamma_2 \in \Gamma$.

Definition (Invariant measure)**.** Let $\alpha : \Gamma \curvearrowright X$ be a group action. Call a collection $\mathcal{M} \subseteq \mathscr{P}(X)$ *α-invariant* if $\gamma \cdot_\alpha A \in \mathcal{M}$ for every $\gamma \in \Gamma$ and $A \in \mathcal{M}$. For a measure space $(X, \mathcal{M}, \mu)$, call the measure μ *α-invariant* if $\mathcal{M}$ is α-invariant and $\mu(\gamma \cdot_\alpha A) = \mu(A)$ for all $\gamma \in \Gamma$ and $A \in \mathcal{M}$.

2. (Completion of measure) Let $(X, \mathcal{M}, \mu)$ be a measure space. Let $\overline{\mathcal{M}}$ denote the collection of the sets of the form $A \cup Z$, where $A \in \mathcal{M}$ and Z is a μ-null set. Prove the following.

(a) $\overline{\mathcal{M}}$ is a σ-algebra.

(b) The map $\overline{\mu} : \overline{\mathcal{M}} \to [0, \infty]$ given by $\overline{\mu}(A \cup Z) = \mu(A)$ is well-defined, i.e. independent of the presentation $A \cup Z$.

(c) $\overline{\mu}$ is a measure, making $(X, \overline{\mathcal{M}}, \overline{\mu})$ a complete measure space, so we call $\overline{\mu}$ the *completion of μ*.

3. (Measuring boxes) Let $\mathcal{C}$ be the collection of all boxes in $\mathbb{R}^d$ and let $\mathcal{A}$ be the algebra of finite unions of boxes in $\mathbb{R}^d$. Define $\rho : \mathcal{C} \to [0, \infty]$ by

$$\rho(I_1 \times I_2 \times \ldots \times I_d) := |I_1| \cdot |I_2| \cdot \ldots \cdot |I_d|.$$

Call a collection $\mathcal{G} \subseteq \mathscr{P}(\mathbb{R}^d)$ a *grid* if $\mathcal{G} = \mathcal{I}_1 \times \ldots \times \mathcal{I}_d$, where each $\mathcal{I}_i$ is a finite collection of pairwise disjoint intervals. In particular, $\mathcal{G}$ is a finite collection of pairwise disjoint boxes.

(a) Prove that if a box B is partitioned[1] into a grid $\mathcal{G}$, then

$$\rho(B) = \sum_{C \in \mathcal{G}} \rho(C).$$

(b) Prove that every finite partition[1] $\mathcal{P}$ of a box B into other boxes can be *refined* to a grid, i.e. there is a grid $\mathcal{G}$ that partitions B and is such that every $C \in \mathcal{G}$ is contained in some $C' \in \mathcal{P}$.

(c) Conclude that if $A \in \mathcal{A}$ admits two finite partitions $\mathcal{P}_0, \mathcal{P}_1$ into boxes, then

$$\sum_{B \in \mathcal{P}_0} \rho(B) = \sum_{C \in \mathcal{P}_1} \rho(C).$$

Thus, there is a unique finitely additive measure $\bar{\rho} : \mathcal{A} \to [0, \infty]$ that extends ρ.

[1] A *partition* $\mathcal{P}$ of a set A is a collections of pairwise disjoint sets that union up to A.

4. (Coin-flip premeasure) Recalling the Cantor space $2^{\mathbb{N}}$ from the previous homework, let $\mathcal{C} \subseteq \mathscr{P}(2^{\mathbb{N}})$ be the collection of all sets of the form V_s (call these sets *cylinders*) and let $\mathcal{A}$ be the algebra of finite unions of cylinders. Define a $\rho : \mathcal{C} \to [0,1]$ by $\rho(V_s) := 2^{-|s|}$ for each $s \in 2^{<\mathbb{N}}$.

(a) Prove that ρ admits a unique extension to a finitely additive probability measure $\bar{\rho} : \mathcal{A} \to [0,1]$. This is called the *fair coin-flip measure.*

Hint. Proceed as in Problem 3, defining the analogous notion of a *grid* as a collection of the form $\{V_t : t \in 2^n, t \supseteq s\}$ for some $s \in 2^m$ with $0 \leqslant m \leqslant n$.

(b) Prove that $\bar{\rho}$ is actually a premeasure on $\mathcal{A}$, i.e. that it is countably-additive.

Hint. The sets V_s are compact, so you only have to recall the following definition of compactness (which, in metric spaces, is equivalent to the one with sequences): every open cover has a finite subcover.

5. Let ρ be a premeasure on an algebra $\mathcal{A}$ on a set X and let $\mu^* : \mathscr{P}(X) \to [0, \infty]$ be the induced premeasure as defined in class (or in Theorem 4.17 of Bass's textbook). Prove that for every $A \in \mathcal{A}$ and every $B \subseteq X$, $\mu^*(B) = \mu^*(B \cap A) + \mu^*(B \cap A^c)$. You may use the fact that μ^* coincides with ρ on $\mathcal{A}$.