Math 540: Real Analysis HOMEWORK 12 **Optional, won't be graded**

Notation and terminology. Below let H denote a Hilbert space.

For any closed linear subspace $M \subseteq H$ and any vector $x \in M$, we have proven in class that there is a unique vector $y \in M$ such that $x - y \in M^{\perp}$. Call this vector y the *projection of* x *on* M and denote it by $\operatorname{proj}_M(x)$. Recall that $\operatorname{proj}_M(x)$ is also the closest element of M to x, i.e. $\|x - y\| = \inf_{y' \in M} \|x - y'\|$. Call $x - y$ the *coprojection of* x *on* M and denote it by $\operatorname{proj}_M^{\perp}(x)$. Observe that $\operatorname{proj}_M^{\perp}(x) = \operatorname{proj}_{M^{\perp}}(x)$.

Problems.

1. Let $\{u_i\}_{i\in I} \subseteq H$ be an orthonormal family and let $M := \overline{\operatorname{Span}\{u_i : i \in I\}}$. Prove that the set $I_x := \{i \in I : \langle x, u_i\rangle \neq 0\}$ is countable and, for any enumeration $I_x = \{i_n\}_{n\in\mathbb{N}}$,

$$\sum_{k<n} \langle x, u_{i_k}\rangle\, u_{i_k} \to \operatorname{proj}_M(x) \text{ as } n \to \infty.$$

In other words,

$$\operatorname{proj}_M(x) = \sum_{i\in I} \langle x, u_i\rangle\, u_i.$$

Conclude that $\|\operatorname{proj}_M(x)\|^2 = \sum_{i\in I} |\langle x, u_i\rangle|^2$.

2. Let H be a Hilbert space and let $\{u_i\}_{i\in I} \subseteq H$ be an orthonormal family of vectors. Prove that the following are equivalent:

(1) $\{u_i\}_{i\in I}$ is an orthonormal basis of H, i.e. $\overline{\operatorname{Span}\{u_i : i \in I\}} = H$.

(2) Every $x \in H$ is equal to $\sum_{i\in I} \langle x, u_i\rangle\, u_i$ (in the sense of Problem 1).

(3) For every $x \in H$, $\|x\|^2 = \sum_{i\in I} |\langle x, u_i\rangle|^2$.

(4) $\{u_i\}_{i\in I}$ is a maximal orthonormal family, i.e. if an $x \in H$ is orthogonal to all u_i then $x = 0$.

3. Prove that a Hilbert space is separable if and only if it admits a countable orthonormal basis.

4. Prove that any two orthonormal bases of a Hilbert space H have the same cardinality.

Hint. Use (2) of Problem 2 together with the fact that $|\mathbb{N} \times I| = |I|$ for any infinite set I.

5. Let H be a Hilbert space and let $\{u_i\}_{i\in I} \subseteq H$ be an orthonormal basis of H. Show that the map $H \to \ell^2(I)$ defined by $x \mapsto \hat{x} := \big(\langle x, u_i\rangle\big)_{i\in I}$ is unitary. In particular, any separable Hilbert space is isomorphic to $\ell^2(\mathbb{N})$.

6. Prove the stronger version of the Lebesgue differentiation theorem (Corollary 10 in the "Lebesgue differentiation" lecture notes.)

Definition. Call a function $f : \mathbb{R} \to \mathbb{R}$ *absolutely continuous* (*in the uniform norm*), if for every $\varepsilon > 0$ there is $\delta > 0$ such that for any finite set $\{(a_i, b_i)\}_{i<n}$ of disjoint intervals,

$$\sum_{i<n} (b_i - a_i) < \delta \implies \sum_{i<n} |f(b_i) - f(a_i)| < \varepsilon.$$

7. Let $f \in L^1_{\text{loc}}(\mathbb{R})$ and $a \in \mathbb{R}$. Define an *antiderivative* $F : \mathbb{R} \to \mathbb{R}$ of f by

$$F(x) := \int_a^x f(t) d\lambda(t).$$

(a) Prove that F is absolutely continuous (in the uniform norm).

(b) Prove that F is differentiable almost everywhere and $F' = f$ a.e.

Definition. Let E be an equivalence relation on a set X. Call a set $A \subseteq X$ *E-invariant* if A is a union of E-classes, equivalently, for every $x \in X$, $x \in A \Rightarrow [x]_E \subseteq A$. Letting Y be a set, call a function $f : X \to Y$ *E-invariant* if f-preimage of every point is E-invariant, equivalently, for any $x_0, x_1 \in X$, $x_0 \, E \, x_1 \Rightarrow f(x_0) = f(x_1)$.

Definition. Call an equivalence relation E on a measure space $(X, \mathcal{M}, \mu)$ *ergodic*[1] (or *μ-ergodic*) if every E-invariant measurable set is either μ-null or μ-conull.

8. Let E be an ergodic equivalence relation on a measure space $(X, \mathcal{M}, \mu)$. Prove that any measurable E-invariant function $f : X \to \mathbb{R}$ is constant a.e.

9. Recall the Vitali equivalence relation E_V on $\mathbb{R}$: $x \, E_V \, y :\Leftrightarrow x - y \in \mathbb{Q}$. Prove that it is ergodic with respect to the Lebesgue measure. In particular, any measurable E_V-invariant function is constant a.e., which gives another proof that choosing a point from each E_V-class results in a nonmeasurable set.

Hint. Use the 99% lemma.

10. Read the definition of the *Cantor–Lebesgue function* on Christopher Hail's note on Cantor–Lebesgue function [link to pdf, also posted on our course webpage] and do Exercise 1.57 of the same note.

[1] Think of *ergodic* as *atomic* in the eyes of the measure.