Math 540: Real Analysis HOMEWORK 11 **Due date: Apr 25 (Tue)**

Exercises from Bass's textbook. 15.26, 15.27, 15.29

Missing condition (in 15.26)**.** The measure space should be assumed to be σ-finite to enable the use of the Tonelli-Fubini theorem.

Remark (on 15.26). The linear operator T defined in 15.26 is called the *Hilbert-Schmidt operator with kernel* $K(x, y)$. For $1 \le p < \infty$, what the exercise claims is that, viewing T as a linear operator on L^p, the operator norm of T is bounded above by M.

1. Follow the steps below to prove the Weierstrass approximation theorem.

Theorem (Weierstrass)**.** *Polynomials are uniformly dense in* $C([a, b])$*.*

Suppose without loss of generality that $[a, b] = [0, 1]$ and let $f : [0, 1] \to \mathbb{R}$ be continuous. For $\alpha > 0$, denote $I_\alpha := [-\alpha, \alpha]$.

(a) Argue that we may assume without loss of generality that $f(0) = f(1) = 0$. Hence we may assume that the domain of f is the entire $\mathbb{R}$ and $f|_{\mathbb{R}\setminus[0,1]} \equiv 0$.

Hint. Subtract a linear polynomial from f.

(b) Let $q_n(x) := c_n(1 - x^2)^n$, where c_n is a positive constant such that $\int_{I_1} q_n(t)dt = 1$. Prove that for any $\delta > 0$, $q_n|_{I_1\setminus I_\delta} \to_u 0$.

Hint. For example, show that $c_n \le \sqrt{n}$.

(c) Show that a convolution of a function with a polynomial is a polynomial and that, for any $x \in [0, 1]$,

$$p_n(x) := f * q_n(x) = \int_{I_1} f(x - t)q_n(t)dt.$$

(d) Prove that $p_n|_{[0,1]} \to_u f|_{[0,1]}$ as $n \to \infty$.

Hint. For $\varepsilon > 0$, use the uniform continuity of f to get a δ. Writing $|p_n(x) - f(x)|$ as an integral over I_1, split I_1 into I_δ and $I_1 \setminus I_\delta$.

2. Let V be a complex inner product space. Prove the so-called *polarization identity*:

$$\langle x, y\rangle = \frac{\|x + y\|^2 - \|x - y\|^2}{4} + i\frac{\|ix - y\|^2 - \|ix + y\|^2}{4}.$$

Remark. In inner product spaces, the norm is defined in terms of the inner product. The polarization identity expresses the inner product in terms of the norm.

3. (Optional) Let V be a normed vector space and suppose that its norm $\|\cdot\|$ satisfies the parallelogram law. Follow the steps below to prove that

$$\langle x, y\rangle := \frac{\|x + y\|^2 - \|x - y\|^2}{4} + i\frac{\|ix - y\|^2 - \|ix + y\|^2}{4}$$

is an inner product on V. For simplicity, you may assume that V is a real normed vector space and $\langle x, y\rangle := \frac{\|x+y\|^2-\|x-y\|^2}{4}$.

(a) Show that $\langle 2x, y\rangle = 2\langle x, y\rangle$.

(b) Using the previous part, show that $\langle x_1 + x_2, y\rangle = \langle x_1, y\rangle + \langle x_2, y\rangle$.

(c) Show that $|\langle x, y\rangle| \leq \|x\|\|y\|$ and deduce that, for each fixed $x, y \in V$, the map $\alpha \mapsto \langle \alpha x, y\rangle : \mathbb{R} \to \mathbb{R}$ is continuous.

(d) Prove that any continuous additive[1] map $f : \mathbb{R} \to \mathbb{R}$ is in fact just a scalar multiplication: $f(x) = f(1) \cdot x$ for all $x \in \mathbb{R}$.

Hint. First show that $f(nx) = nf(x)$ for any $n \in \mathbb{Z}, x \in \mathbb{R}$, and deduce $f\left(\frac{1}{n}x\right) = \frac{1}{n}f(x)$ for $n \in \mathbb{Z} \setminus \{0\}$. Deduce further that, for any $q \in \mathbb{Q}$, $f(qx) = qf(x)$ and use continuity.

(e) Deduce that for all $\alpha \in \mathbb{R}$, $\langle \alpha x, y\rangle = \alpha\langle x, y\rangle$.

4. Let U be a linear subspace of an inner product space V and let $x \in V, y \in U, z \in U^{\perp}$. Show that $x = y + z$ if and only if y, z are the closest vectors to x in $U, U^{\perp}$, respectively.

5. Let V be an inner product space.

Definition. Call a collection $S \subseteq V$ of vectors *orthogonal* if the vectors in it are pairwise orthogonal. Call S *orthonormal* if it is orthogonal and each vector in it is normal.

Let I be a finite set and let $S := \{v_i\}_{i \in I} \subseteq V$ be orthonormal. For $x \in V$, define

$$\mathrm{proj}_S(x) := \sum_{i \in I} \mathrm{proj}_{v_i}(x) = \sum_{i \in I} \langle x, v_i\rangle\, v_i$$

and put $\mathrm{proj}_S^{\perp}(x) := x - \mathrm{proj}_S(x)$.

(a) Prove that $\mathrm{proj}_S^{\perp} \perp U := \mathrm{Span}(S)$ and that $\mathrm{Span}(\{x\} \cup U) = \mathrm{Span}(\mathrm{proj}_S^{\perp}(x) \cup U)$.

(b) (Gram–Schmidt process) Given a countable collection of vectors $\{u_n\}_{n \in \mathbb{N}}$ inductively define an orthogonal collection $\{v_n\}_{n \in \mathbb{N}}$ such that each vector in it is either normal or 0 and, for every $N \in \mathbb{N}$, $\mathrm{Span}(\{u_n\}_{n<N}) = \mathrm{Span}(\{v_n\}_{n<N})$.

[1] This means $f(r + s) = f(r) + f(s)$