Math 540: Real Analysis HOMEWORK 10 **Due date: Apr 18 (Tue)**

Exercises from Bass's textbook. 15.8, 15.9, 15.13, 15.17, 15.21

1. Let μ be a finite Borel measure on $\mathbb{R}$ (i.e. defined on $\mathcal{B}(\mathbb{R})$) with the property that, for every bounded Borel function $f : \mathbb{R} \to \mathbb{R}$,

$$f(x) = \int_{\mathbb{R}} f(x+t) d\mu(t).$$

(a) Prove that $\mu(\{0\}) = 1$ and deduce that $\mu(\mathbb{R} \setminus \{0\}) = 0$, so $\mu = \delta_0$, where δ_0 is the so-called *Dirac measure* (also known as the *point-mass* and *delta function*) at 0.

(b) Does $f \mapsto \int_{\mathbb{R}} f d\delta_0$ define a linear functional on $L^p(\mathbb{R}, \mathcal{B}(\mathbb{R}), \lambda)$ for some $1 \leq p \leq \infty$?

2. In class we proved that $\partial_i(f * \varphi) = f * \partial_i \varphi$, which involved moving a limit inside an integral; this exercise is a more general statement.

Let $(X, \mathcal{M}, \mu)$ be a measure space and let $f : X \times [a, b] \to \mathbb{R}$, $-\infty < a < b < \infty$, be such that, for each $t \in [a, b]$, $f^t \in L^1(X, \mu)$. Let $F(t) := \int_X f(x, t) d\mu(x)$.

(a) Let $t_0 \in [a, b]$ and suppose that $\lim\limits_{t \to t_0} f(x, t)$ exists for a.e. $x \in X$.

(i) Prove that the function $x \mapsto \lim\limits_{t \to t_0} f(x, t)$ is $\overline{\mathcal{M}}$-measurable.

(ii) Suppose that there is $g \in L^1$ such that, for a.e. $x \in X$, $|f(x, t)| \leq g(x)$ for every $t \in [a, b]$. Prove that

$$\lim_{t \to t_0} \int_X f(x, t) d\mu(x) = \int_X \lim_{t \to t_0} f(x, t) d\mu(x).$$

Deduce that if f_x is continuous for at t_0 for a.e. $x \in X$, then F is continuous at t_0.

(b) Suppose that, for a.e. $x \in X$, $\frac{\partial f}{\partial t}(x, t_0)$ exists for every $t_0 \in [a, b]$.

(i) Prove that, for every $t_0 \in [a, b]$, the function $x \mapsto \frac{\partial f}{\partial t}(x, t_0)$ is $\overline{\mathcal{M}}$-measurable.

(ii) Suppose that there is $g \in L^1$ such that, for a.e. $x \in X$, $\left|\frac{\partial f}{\partial t}(x, t)\right| \leq g(x)$ for every $t_0 \in [a, b]$. Prove that F is differentiable and $F'(t_0) = \int_X \frac{\partial f}{\partial t}(x, t_0) d\mu(x)$.

3. Let $f \in L^1(\mathbb{R}^d, \lambda)$ and prove that

(a) $\int_{\mathbb{R}^d} f(x + y) dx = \int_{\mathbb{R}^d} f(x) dx$

(b) $\int_{\mathbb{R}^d} f(-x) dx = \int_{\mathbb{R}^d} f(x) dx$

(c) $\int_{\mathbb{R}^d} f(\alpha x) dx = \alpha^{-d} \int_{\mathbb{R}^d} f(x) dx$, for any $\alpha > 0$.

Caution. Note that it's α^{-d} and not α^d.