Math 540: Real Analysis HOMEWORK 1 **Due date: Jan 24 (Tue)**

1. (a) Prove that if $f : [0,1] \to \mathbb{R}$ is a bounded Riemann-integrable function and $g : \mathbb{R} \to \mathbb{R}$ is continuous, then $g \circ f$ is Riemann-integrable.

(b) Recall that a set $A \subseteq [0,1]$ is called *Riemann-measurable* if its indicator function $\mathbf{1}_A$ is Riemann-integrable. Using (a) and the fact that Riemann-integrable functions are closed under addition, prove that the Riemann-measurable subsets of $[0,1]$ form an algebra on $[0,1]$.

Hint. $\min(x,y) = \frac{x+y}{2} - \frac{|x-y|}{2}$.

2. (Introducing Cantor space) Let $2^{\mathbb{N}}$ denote the set of all infinite binary sequences and let $2^{<\mathbb{N}}$ denote the set of all finite binary sequences (including the empty sequence $\emptyset$). For $s \in 2^{<\mathbb{N}}$, let $|s|$ denote its length. For example, $1101 \in 2^{<\mathbb{N}}$ and $|1101| = 4$. For $x \in 2^{\mathbb{N}}$ and $n \in \mathbb{N}$, let $x\lfloor_n$ denote the finite initial segment of x of length n. For $s \in 2^{<\mathbb{N}}$, say that x *extends* s, and write $x \supseteq s$, if $x\lfloor_{|s|} = s$. For each $s \in 2^{<\mathbb{N}}$, define

$$V_s := \left\{ x \in 2^{\mathbb{N}} : x \supseteq s \right\}.$$

Thinking of $2^{<\mathbb{N}}$ as the binary tree and $2^{\mathbb{N}}$ as the set of infinite branches through it, V_s is the set of all infinite branches of the subtree lying under[1] s.

(a) Prove that the finite unions of the sets V_s form an algebra on $2^{\mathbb{N}}$.

(b) Define $d : 2^{\mathbb{N}} \times 2^{\mathbb{N}} \to [0,\infty)$ by $(x,y) \mapsto 2^{-\Delta(x,y)}$, where $\Delta(x,y)$ is the largest n such that $x\lfloor_n = y\lfloor_n$. Show that d is a metric on $2^{\mathbb{N}}$, whose open balls are exactly the sets V_s. The metric space $(2^{\mathbb{N}}, d)$ is called the *Cantor space* and denoted by $\mathcal{C}$.

(c) Conclude that the sets V_s are clopen.

(d) Prove that $2^{\mathbb{N}}$ is compact.

Hint. Recall/learn the proof of König's lemma[2], which is the same as the divide-and-choose proof of the Bolzano-Weierstrass theorem.

3. Let X be a metric space. Recall that a set $A \subseteq X$ is called *Baire-measurable* if $A = U \triangle M$, where U is open, M is meager (i.e. is a countable union of nowhere dense sets), and $\triangle$ denotes the symmetric difference. Prove that Baire-measurable sets form a σ-algebra.

Hint. Showing closure under complements boils down to closed sets being Baire-measurable.

4. (a) Let $(X, \mathcal{A}), (Y, \mathcal{B})$ be measurable spaces and let $\mathcal{C}$ be a generating collection for $\mathcal{B}$, i.e. $\sigma(\mathcal{C}) = \mathcal{B}$. Prove that, for any function $f : X \to Y$, if $f^{-1}(C) \in \mathcal{A}$ for all $C \in \mathcal{C}$, then f is measurable, i.e. $f^{-1}(B) \in \mathcal{A}$ for all $B \in \mathcal{B}$.

(b) Conclude that for metric spaces X, Y, a function $f : X \to Y$ is Borel if and only if f-preimages of open sets are Borel. In particular, continuous functions are Borel.

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[1] My trees grow downward; if yours grow upward, replace "under" with "above".

[2] König's Lemma: Every finitely branching infinite tree has an infinite branch.