Exercises from Kaplansky's book.

Sec 4.2: 1, 2, 7

Sec 4.3: 3

Definition. For a linear order $(A, <)$, call a subset $B \subseteq A$ *dense* if, for every pair a_1, a_2 in A with $a_1 < a_2$, there is $b \in B$ with $a_1 < b < a_2$.

Below, when $\mathbb{R}$ and $2^{\mathbb{N}}$ are used as metric spaces, they are assumed to be equipped with their usual metrics, unless stated otherwise.

1. Show that

(a) $\mathbb{Q}$ is dense in $(\mathbb{Q}, <)$.

(b) $\mathbb{Q}$ is dense in $(\mathbb{R}, <)$.

HINT: One last time, reals are Dedekind cuts.

(c) $\mathbb{R} \setminus \mathbb{Q}$ is dense in $(\mathbb{R}, <)$.

HINT: $q + \sqrt{2}$ is irrational for any $q \in \mathbb{Q}$.

(d) Neither $\mathbb{Q}$ nor $\mathbb{R} \setminus \mathbb{Q}$ is open in $\mathbb{R}$, and hence neither is closed.

2. Let (X, d) be a metric space and $A \subseteq X$. Prove:

(a) For an open set U, $U \cap A = \emptyset \implies U \cap \overline{A} = \emptyset$.

(b) The boundary ∂A is a closed set.

3. For each of the following, determine the boundary and closure of the set A in the metric space (X, d). Prove your answers.

(a) $X := \mathbb{R}$, $A := \{q \in \mathbb{Q} : q < 0 \text{ or } q^2 \leq 2\}$.

(b) $X := \mathbb{R}$, $A := \left\{\frac{1}{n} : n \in \mathbb{N} \setminus \{0\}\right\}$.

(c) $X := (0, 1) \cup [2, 3]$, $A := (0, 1) \cup \{2\}$.

(d) $X := (0, 1) \cup [2, 3]$, $A := (2, 3)$.

(e) $X := 2^{\mathbb{N}}$, $A :=$ the set of eventually 0 sequences, i.e. $A := \left\{x \in 2^{\mathbb{N}} : \forall^{\infty} n \in \mathbb{N}\ x(n) = 0\right\}$.